Qualifying Examination
LOGIC
Fall 2017

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Prove that there is no formula $\varphi(v, u)$ so that $(\mathbb{N}; 0, 1, +) \models \varphi[x, y]$ iff x divides y.

Problem 2. Let $\mathcal{L}$ be the language of graphs; the only non-logical symbol of the language is a binary relation symbol, E.

(2a) For which $n < \omega$ is there a finitely axiomatizable theory T_n in $\mathcal{L}$, so that the models of T_n are exactly the n-colorable graphs?

(2b) Construct an ω-saturated model $\mathbf{A} = (A; E_{\mathbf{A}})$ of $\mathcal{L}$, so that for every $n < \omega$ there is a definable (from parameters) set $D_n \subseteq A$, so that $(D_n; E_{\mathbf{A}} \restriction D_n)$ forms a graph which is $n+1$-colorable but not n-colorable.

Problem 3. Say that a formula $\varphi(v_1, \ldots, v_l, u)$ in the language of set theory *defines* a set x from parameters $a_1, \ldots, a_l$, and x is *definable* from $a_1, \ldots, a_l$, if $\varphi(a_1, \ldots, a_l, x)$ holds (in V) and for every $y \neq x$, $\varphi(a_1, \ldots, a_l, y)$ fails.

(3a) Suppose there is a model of ZFC. Prove there is a model of ZFC in which every set is definable from ordinal parameters.

(3b) Prove that if x is definable from ordinal parameters, then it is definable from ordinal parameters by a Σ_2 formula.

Problem 4. Assume AC. Assume that $\aleph_2^{\aleph_1} = \aleph_2$ and $(\forall \alpha < \aleph_2)\alpha^{\aleph_0} = \aleph_1$ (under AC these are consequences of the GCH).

(4a) Prove that there is a sequence $\langle A_\xi \mid \xi < \omega_2 \rangle$ of bounded subsets of $\aleph_2$ so that for every $A \subseteq \aleph_2$, there is a club of $\gamma < \aleph_2$ so that $(\forall \beta < \gamma) A \cap \beta \in \{A_\xi \mid \xi < \gamma\}$.

(4b) Prove that there is a sequence $\langle F_\alpha \mid \alpha < \omega_2, \operatorname{cof}(\alpha) = \omega \rangle$ so that each F_α is a family of subsets of α, $|F_\alpha| \leq \aleph_1$, and for each $A \subseteq \aleph_2$, $\{\alpha < \aleph_2 \mid A \cap \alpha \in F_\alpha\}$ is stationary.

Problem 5. Let φ_i be the ith partial recursive function from ω to ω. Let $T = \{i \colon \varphi_i \text{ is total}\}$. Show that every Π_2 subset of ω is many-one reducible to T.

Problem 6. Let W_i be the ith computably enumerable set. Say that i is a *minimal index* if for all $j < i$, $W_j \neq W_i$.

(6a) Show that $\{i : i \text{ is a minimal index}\}$ contains no infinite computably enumerable subset.

(6b) Show that there are only finitely many i such that PA proves "i is a minimal index".

Problem 7. Let $\mathrm{Prov}_{\mathsf{PA}}(n)$ be the formula of arithmetic asserting that there is a proof of the formula with Gödel number n from PA. Let $\ulcorner\varphi\urcorner$ indicate the (numeral of the) Gödel number of the formula φ. You may use that the provability predicate satisfies the following three properties for all sentences φ and ψ:

(1) If $\mathsf{PA} \vdash \varphi$, then $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$.
(2) If $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \to \psi \urcorner)$, then $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \psi \urcorner)$.
(3) $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \urcorner)$.

(7a) Fix a sentence φ. Find a sentence θ so that $\mathsf{PA} \vdash \theta \leftrightarrow (\mathrm{Prov}_{\mathsf{PA}}(\ulcorner \theta \urcorner) \to \varphi)$.

(7b) Assuming $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$, show that θ is provable from PA. Use this to derive Löb's theorem that if $\mathsf{PA} \vdash \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$, then $\mathsf{PA} \vdash \varphi$

(7c) Give a sentence φ so that PA does not prove $\varphi \to \mathrm{Prov}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$.

Problem 8. A structure $\mathcal{A}$ is computable if its universe A is a computable subset of ω, and its functions, relations, and constants are uniformly computable. (Or equivalently, the atomic diagram of A is computable.) Show there is no computable model of ZFC.