Qualifying Examination
LOGIC
Fall 2018

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1. Let $\mathcal{B}$ be an infinite *atomless* boolean algebra, viewed as a first-order structure in the language $\mathcal{L} = (0, 1, \wedge, \vee, \neg)$ (i.e., it is a Boolean algebra such that for any element a there is some element b such that $a \wedge b \neq 0$ and $a \wedge \neg b \neq 0$). Let $T := \mathrm{Th}\,(\mathcal{B})$ be its complete theory.

(1a) Show that T eliminates quantifiers.

(1b) Show that T is $\aleph_0$-categorical.

(1c) Show that for $\kappa > \aleph_0$, T is not κ-categorical.

Problem 2. Let T be a first-order theory in a language $\mathcal{L}$ admitting an infinite model. Show that for every cardinal $\kappa \geq |\mathcal{L}| + \aleph_0$ there is a model $\mathcal{N} \models T$ of cardinality κ with 2^κ automorphisms.

Problem 3. Determine which of the following is true for arbitrary Σ_1 sentences φ (with "$\mathsf{PA} \vdash \varphi$" expressed in the language of arithmetic in the standard way for (3b) and (3d)).

(3a) $\varphi \to \mathsf{PA} \vdash \varphi$.

(3b) $\mathsf{PA} \vdash (\varphi \to \mathsf{PA} \vdash \varphi)$.

(3c) $(\mathsf{PA} \vdash \varphi) \to \varphi$.

(3d) $\mathsf{PA} \vdash ((\mathsf{PA} \vdash \varphi) \to \varphi)$.

(3e) $\neg\varphi \to \mathsf{PA} \vdash \neg\varphi$.

Problem 4. **(4a)** Prove that there are recursively inseparable r.e. sets. That is, show that there exist recursively enumerable sets $A, B \subseteq \mathbb{N}$ with $A \cap B = \emptyset$ such that there is no recursive set $C \subseteq \mathbb{N}$ with $A \subseteq C$ and $C \cap B = \emptyset$.

(4b) Show that there is a partial recursive unary function which cannot be extended to a total recursive function.

Problem 5. Prove that there is no recursive non-standard model of PA. That is, show that there is no model $\mathcal{M} = (\mathbb{N}; 0^*, S^*, +^*, \times^*, <^*)$ with universe $\mathbb{N}$, in the language of arithmetic, so that $\mathcal{M}$ is (isomorphic to) a non-standard model of PA, and S^*, $+^*$, $\times^*$, and $<^*$ are recursive. You may use the standard coding results in PA without proof. For example this includes the fact that for every formula $\varphi(v)$, $\mathsf{PA} \vdash (\forall n)(\exists k, m)(\forall i < n)(k \bmod (m \cdot i + 1) = 0 \leftrightarrow \varphi(i))$.

Problem 6. Suppose $\kappa^{<\kappa} = \kappa$. For full credit, prove that there is a κ^+-Aronszajn tree. Or prove that there is an $\aleph_1$-Aronszajn tree for half credit. A κ^+-Aronszajn tree is a tree of height κ^+, whose levels each have size κ, with no cofinal branch.

Problem 7. Work in the constructible universe L. Fix a regular infinite cardinal κ. For each $\alpha \in [\kappa, \kappa^+)$ let $\beta(\alpha)$ be least so that in $L_{\beta(\alpha)+1}$ there is a surjection of κ onto α. Let $S \subseteq \kappa^+$ and let $D = \{\alpha < \kappa^+ \mid S \cap \alpha \in L_{\beta(\alpha)} \wedge L_{\beta(\alpha)} \models S \cap \alpha \text{ is stationary}\}$. Prove that if D is stationary then S is stationary.

Problem 8. A function $f\colon {}^{<\omega}2 \to {}^{<\omega}2$ *codes* a function $f^*\colon {}^{\omega}2 \to {}^{\omega}2$ if for every $x \in {}^{\omega}2$, $f^*(x) = \bigcup_{n<\omega} f(x \restriction n)$. Fix some recursive bijection $s \mapsto \ulcorner s \urcorner$ of ${}^{<\omega}2$ onto ω. Let φ_e be the standard enumeration of recursive functions on ω, and let φ'_e be the induced functions on ${}^{<\omega}2$, meaning that $\ulcorner \varphi'_e(s) \urcorner = \varphi_e(\ulcorner s \urcorner)$. Classify the set $A = \{e \mid \varphi'_e \text{ codes a function on } {}^{\omega}2\}$ in the arithmetical hierarchy.