Qualifying Examination
LOGIC
Fall 2019

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Let $\mathcal{L}$ be a finite language.

(1a) Let Σ be a decidable set of $\mathcal{L}$-sentences such that for each $\sigma \in \Sigma$ we have $\models \sigma$ if and only if $\mathbf{A} \models \sigma$ for every *finite* $\mathcal{L}$-structure $\mathbf{A}$. Show that then $\{\sigma \in \Sigma : \models \sigma\}$ is decidable.

(1b) Suppose that $\mathcal{L}$ only consists of function symbols. Show that then the set of universal $\mathcal{L}$-sentences σ such that $\models \sigma$ is decidable.

(1c) Suppose that $\mathcal{L}$ contains a single binary function symbol. Show that then the set of all $\mathcal{L}$-sentences σ such that $\models \sigma$ is undecidable.

(In contrast to (c), it is still true [Ehrenfeucht, Rabin] that if $\mathcal{L}$ only contains a single unary function symbol, then the set of all $\mathcal{L}$-sentences σ such that $\models \sigma$ is decidable.)

Problem 2. Let $\mathcal{L}$ be a language and Σ be a set of $\mathcal{L}$-sentences whose class of models is closed under taking substructures and (finite) direct products, and let Σ_∞ be a set of $\mathcal{L}$-sentences whose models are the infinite models of Σ. Suppose for each $\mathcal{L}$-formula $\varphi(x)$ there is an $\exists$-formula $\varphi^*(x)$ with $\Sigma_\infty \models \forall x(\varphi \leftrightarrow \varphi^*)$. Show that φ^* can even always be taken to be quantifier-free.

Problem 3. In this problem $n \geq 1$.

(3a) Let $\mathbf{M} = (M; \dots)$ be a relational structure. Suppose there is a well-ordering of M which is definable in $\mathbf{M}$ by a Σ_1-formula in the language of $\mathbf{M}$. We say that $s \in M$ is Π_n-*definable* if there is a Π_n-formula which holds in $\mathbf{M}$ of s and no other elements. Show that the set $S \subseteq M$ consisting of all Π_n-definable elements is (the underlying set of) a Σ_n-elementary substructure of $\mathbf{M}$.

(3b) Let L_θ be a level of L satisfying a large enough finite fragment of ZFC. Show that the set of Π_n-definable elements of L_θ is a Σ_n-elementary substructure of L_θ.

Problem 4. Let Γ be a finite fragment of ZFC. Prove that there is α so that $L_\alpha \models \Gamma$ and L_α is countable in $L_{\alpha+1}$. (Hint: some of the earlier problems in this exam may be useful.)

Problem 5. Work in ZF plus Countable Choice (only). Let κ be a cardinal of uncountable cofinality. Prove that at least one of the following holds: (i) Every stationary subset of κ has two disjoint stationary subsets; (ii) There is a countably complete non-principal ultrafilter on κ.

Problem 6. Let $(\varphi_e)_{e\in\mathbb{N}}$ be the standard enumeration of the partial recursive functions. By $f^{(n)}$ we mean the nth iterate of a partial function $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$, that is $f^{(0)} =$ identity function on $\mathbb{N}$ and $f^{(n+1)} = f \circ f^{(n)}$. Classify the following set in the arithmetical hierarchy:

$$A = \big\{e \in \mathbb{N} : \varphi_e^{(5)} = \varphi_e^{(3)}\big\}.$$

Here equality is meant in the strong sense that the left side is defined if and only if the right side is defined, and in this case the two are equal.

Problem 7. Recall that $\mathcal{K} = \emptyset' = \big\{e \in \mathbb{N} : \phi_e(e)\downarrow\big\}$. Provability is formalized using Gödel codes in the usual way, Δe is the term denoting $e \in \mathbb{N}$, and $t \in \mathcal{K}$ for a term t is the natural formalization of the statement that the interpretation of t belongs to $\mathcal{K}$. Determine which of the following is true:

(7a) If $e \in \mathcal{K}$, then $\mathsf{PA} \vdash \Delta e \in \mathcal{K}$.

(7b) If $e \notin \mathcal{K}$, then $\mathsf{PA} \vdash \Delta e \notin \mathcal{K}$.

(7c) If $\mathsf{PA} \vdash \Delta e \notin \mathcal{K}$, then $e \notin \mathcal{K}$.

Problem 8. Let $\mathbf{Proof}(v, u)$ numeralwise express in ZFC the relation

$$\mathbf{Proof}(x, y) \iff x \text{ is the code of sentence } \theta \text{ and } y \text{ is the code of a proof of } \theta \text{ in } \mathsf{ZFC},$$

and set $\mathbf{Pf}(v) :\equiv (\exists u)\mathbf{Proof}(v, u)$. Write $\ulcorner\sigma\urcorner$ for the numeral of the Gödel number of the sentence σ, in some standard Gödel numbering. Show that there is a unique sentence σ, up to provable equivalence in ZFC, such that $\mathsf{ZFC} \vdash \sigma \leftrightarrow \mathbf{Pf}(\ulcorner\neg\sigma\urcorner)$.