Qualifying Examination
LOGIC
Winter 2019

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1

Let T be a theory in a language $\mathcal{L} = \{E_i(x, y) : i \in \omega\}$ expressing the following:

(1) E_i is an equivalence relation with each equivalence class infinite, for each $i \in \omega$.
(2) E_0 has two classes.
(3) E_{i+1} refines E_i, and every class of E_i is given by the disjoint union of two classes of E_{i+1}.

Answer the following questions, with proofs.

(a) Show that T is a complete theory eliminating quantifiers.
(b) What is the number of 1-types over $\emptyset$? What is the number of 2-types over $\emptyset$?
(c) What is the number of countable models of T?
(d) Given an infinite cardinal κ, what is the number of models of T of size κ?

Problem 2

Let T be a first-order theory in a countable language $\mathcal{L}$ admitting an infinite model. Show that for every cardinal $\kappa \geq \aleph_0$ there is a model $\mathcal{N} \models T$ of cardinality κ such that: for every $A \subseteq N$, there are at most $|A| + \aleph_0$ types from $S_1^{\mathcal{N}}(A)$ realized in $\mathcal{N}$.

Problem 3

Let Δ be a sufficiently large finite fragment Δ of ZFC. Let $\alpha \leq \beta$, and suppose that $\mathcal{P}(\alpha) \cap (L_{\beta+1} - L_\beta) \neq \emptyset$. Let $\gamma \geq \beta + 1$ be least so that $L_\gamma \models \Delta$. Prove that there is a finite $p \subseteq L_\gamma$ so that every element of L_γ is definable over L_γ from parameters in $\alpha \cup p$. (In particular $|L_\gamma| = |\alpha|$.)

Problem 4

For each sentence σ in the language of arithmetic, let $F(\sigma)$ be the set of consequences of σ, namely $\{\varphi : \sigma \vdash \varphi\}$. Let $A := \{\ulcorner\sigma\urcorner : F(\sigma) \text{ is decidable}\}$, where $\ulcorner - \urcorner$ is the standard Gödel numbering of sentences. Is A recursive? Justify your answer.

Problem 5

Suppose κ is an infinite cardinal and φ is a Σ_2 sentence. Show that if $H_\kappa \models \varphi$, then $V \models \varphi$.

Problem 6

An embedding of a partial order $\leq_P$ into a partial order $\leq_Q$ is an injection $f \colon \operatorname{dom}(P) \to \operatorname{dom}(Q)$ so that $x \leq_P y$ iff $f(x) \leq_Q f(y)$.

(a) Let $\mathcal{F}$ be the set of finite partial orders on subsets of $\mathbb{N}$. Prove that there is a computable partial order $\leq_U$ on $\mathbb{N}$, and a set E of embeddings of orders in $\mathcal{F}$ into U, so that $\emptyset \in E$ and so that for every $P, Q \in \mathcal{F}$ with $Q \restriction \mathrm{dom}(P) = P$, and every $\pi\colon P \to U$ in E, there is $\sigma\colon Q \to U$ in E extending π.

(b) Show there is a computable partial order $\leq_P$ on $\mathbb{N}$ so that for any countable partial order $\leq_Q$ on $\mathbb{N}$, there is an embedding of $\leq_Q$ into $\leq_P$.

Problem 7

Suppose $A \subseteq \mathbb{N}$ is such that for all $B \subseteq \mathbb{N}$, $B \leq_m A$ if and only if $B \leq_1 A$. Show that A is computably isomorphic to a subset of $\mathbb{N} \times \mathbb{N}$ of the form $C \times \mathbb{N}$ for some $C \subseteq \mathbb{N}$.

Problem 8

Show that $A \in 2^{\mathbb{N}}$ is Δ^0_n iff there is a computable function $f(x, y_1, y_2, \ldots, y_{n-1})$ from $\mathbb{N}$ to $\{0, 1\}$ such that $A(x) = \lim_{y_1} \lim_{y_2} \ldots \lim_{y_{n-1}} f(x, y_1, y_2, \ldots, y_{n-1})$. If you use Shoenfield's limit lemma, you must prove it.