Qualifying Exam, Fall 2000

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

MA: Do any 5 problems.

Ph.D.: Do 5 problems and only 3 of them from 1, 2, 3, and 4.

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[1] (a) Derive an expression for the truncation error of the standard second order difference approximation to $\frac{d^2u}{dx^2}$,

$$\frac{d^2u}{dx^2} \simeq \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}$$

where u_n is the value of a function u at the nth point of a grid with mesh size h.

(b) Using the result in (a), derive the order of the local truncation error of the scheme

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = f_n + \frac{h^2}{12}\left(\frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}\right)$$

when used to solve the differential equation

$$\frac{d^2u}{dx^2} = f \qquad u(a) = u(b) = 0$$

[2] Consider using Newton's method to find the root of the polynomial

$$p(x) = (x-1)^2 = x^2 - 2x + 1$$

(a) Does the Newton iteration converge for all initial guesses? Justify your answer.

(b) When it converges, what is the rate of convergence? Justify your answer.

[3] Let A be a symmetric positive definite matrix. At the end of the first step of the LU factorization of A without pivoting, we have

$$A^{(1)} = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \dots & a_{1n} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right)$$

(a) Prove that A' is also symmetric and positive definite.

(b) Using the result from (a), prove that the LU factorization of a symmetric positive definite matrix obtained without pivoting always exists.

[4] Euler's method for solving $\frac{dy}{dt} = f(y)$ is given by

$$y_{n+1} = y_n + dt\ f(y_n) \qquad n \geq 1$$

Consider two methods of using this scheme to advance from y_n to y_{n+1}; $y^{(1)}$ uses a step size of dt, while $y^{(2)}$ uses two steps of size $dt/2$ (see Figure 1).

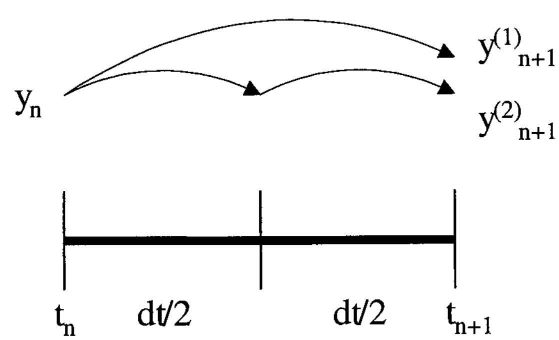

Figure 1.

(a) Let $y^{(*)}_{n+1} = \alpha\ y^{(1)}_{n+1} + \beta y^{(2)}_{n+1}$, how should the values of α and β be chosen so that $y^{(*)}_{n+1}$ is a more accurate solution to the differential equation than either $y^{(1)}_{n+1}$, or $y^{(2)}_{n+1}$? Justify your answer.

(b) For your choice of α and β what is the order of the local truncation error associated with the scheme that advances the solution using $y^{(*)}_{n+1} = \alpha\ y^{(1)}_{n+1} + \beta y^{(2)}_{n+1}$? What is the order of the global error of the method?

[5] Consider the initial value problem

$$\begin{aligned} u_t &= v_x \\ v_t &= 0 \end{aligned}$$

to be solved for $0 \le x \le 1$, $t \ge 0$, with initial and boundary and conditions,

$$\begin{aligned} u(x,0) &= \phi(x) & u(1,t) &= u(0,t) \\ v(x,0) &= \psi(x) & v(1,t) &= v(0,t) \end{aligned} \qquad (4.1)$$

where $\phi(x)$ and $\psi(x)$ are smooth and periodic functions.

(a) Can you write a stable, convergent finite difference scheme for this problem? Explain your answer and give an example of such a scheme if one exists.

(b) Consider the related system

$$\begin{aligned} u_t &= v_x \\ v_t &= \left(\frac{1}{100}\right) u_x \end{aligned}$$

with initial and boundary conditions (4.1). Can you write a stable, convergent finite difference scheme for this problem? Explain your answer and give an example of such a scheme if one exists.

[6] Consider the differential equation

$$u_t = u_{xx} + cu \qquad c < 0$$

with smooth initial data $u_0(x) = u(x,0)$ and $u_0(x)$, $u(x,t)$ periodic with period 1 in x.

(a) Show that the solution decays in time for any initial data.

(b) Construct a stable convergent finite difference scheme whose solutions are second order accurate in space and time and exhibit a similar decay in time. Justify your statements.

[7](a)Derive a variational formulation of the convection-diffusion problem,

$$-\Delta u + a \cdot \nabla u + bu = f(x,y) \qquad 0 < x < 1, \qquad 0 < y < 1$$

$$u = c(x,y) \qquad x = 0, 1 \;\; 0 \le y \le 1$$

$$\frac{\partial u}{\partial \vec{n}} = d(x,y) \qquad 0 < x < 1 \;\; y = 0, 1$$

where a, b, c, d, and f are smooth functions.

(b) Let V_h be an appropriate finite element space (i.e. a space of functions with the requisite approximation properties). Show that the corresponding finite element approximation converges for $b > 0$. What happens when $b = 0$?