Qualifying Exam, Spring 2001
NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM
ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

MA: Do any 5 problems.
Ph.D.: Do 5 problems and only 3 of them from 1, 2, 3 and 4.

---

[1] Consider the three point Legendre-Gauss integration formula

$$\int_{-1}^{1} f(x)dx \cong \tfrac{5}{9}\, f(-\tfrac{\sqrt{15}}{9}) + \tfrac{8}{9}\, f(0) + \tfrac{5}{9}\, f(\tfrac{\sqrt{15}}{9})$$

with the corresponding error expression

$$\int_{-1}^{1} f(x)dx - \left[\tfrac{5}{9}\, f(-\tfrac{\sqrt{15}}{9}) + \tfrac{8}{9}\, f(0) + \tfrac{5}{9}\, f(\tfrac{\sqrt{15}}{9})\right] = \tfrac{-f^{(iv)}(\xi)}{15750} \qquad (1)$$

for some point, ξ, $|\xi| < 1$.

(a) Derive the nodes and weights for the three point Legendre-Gauss quadrature for functions defined over the interval [a,b].

(b) Derive an error expression similar to (1) for the formula you obtain in (a).

[2] Consider the fixed point iteration

$$x_{n+1} = F(x_n)$$

(a) Assuming that the fixed point iteration converges, *explicitly derive* the conditions on F that ensure a second order rate of convergence.

(b) Using your result from (b), derive the condition on $\varphi(x)$ so that the iteration

$$x_{n+1} = x_n + \varphi(x_n) f(x_n)$$

will have a second order rate of convergence to a root α of the problem $f(x) = 0$.

[3] Consider the difference approximation

$$\left.\frac{du}{dx}\right|_0 \approx p_{-1}\, u(-h) + p_0\, u(0) + p_1\, u(h) \tag{2}$$

(a) Show that for any choice of p_0, the choice of

$$\begin{aligned} p_{-1} &= -\frac{1}{2h} - \frac{p_0}{2} \\ p_1 &= \frac{1}{2h} - \frac{p_0}{2} \end{aligned}$$

yields a formula that is exact for all functions of the form $u(x) = a + bx$ with a and b arbitrary constants.

(b) How should p_0 in (2) be chosen so that the formula is exact for all functions of the form $u(x) = a + bx + cx^2$ with a, b and c arbitrary constants. What is the order of accuracy of the resulting difference formula?

(c) How should p_0 in (2) be chosen so that the formula is exact for all functions of the form $u(x) = a + bx + c\,e^{-\gamma x}$ with $a, b,$ c arbitrary constants and γ a fixed constant. (You are deriving the so called "exponential differencing" formulas.) What is the order of accuracy of the resulting difference formula?

[4] Consider the system of ordinary differential equations,

$$y' = Ay + f(y),$$

$$y = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -10^4 \end{pmatrix}$$

$$f = \begin{pmatrix} \sin(u+v) \\ \cos(u-v) \end{pmatrix}.$$

The semi-implicit scheme,

$$y_{n+1} = y_n + h(Ay_{n+1} + f(y_n))$$

is applied to the system.

a) Determine the order of accuracy of the scheme.

b) Determine its stability as $h \to 0$, and its region of absolute stability.

c) Discuss the potential advantages of this scheme above as compared to the explicit and implicit Euler schemes.

[5] Consider the heat equation

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= \nabla \cdot a(x)\nabla u, \quad x \in \Omega \subset \mathbb{R}^2,\ t > 0,\ a \geq a_0 > 0 \\ \tfrac{\partial u}{\partial n} + bu &= f(x), \quad x \in \partial\Omega,\ t > 0 \\ u(x,0) &= u_0(x), \quad x \in \Omega \end{aligned}$$

a) Give a weak formulation of the problem.

b) Describe how to use the Galerkin method together with Crank-Nicolson discretization in time to obtain a numerical method based on piecewise linear elements.

c) Show that the matrices that need to be inverted at each time step are nonsingular for $b = 0$.

[6] Consider the equation

$$u_{tt} = u_{xx}$$

to be solved for

$$0 \leq x \leq 1, \quad t > 0$$

with intial data

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned}$$

u_0, u_1 smooth and vanishing near $x = 0,\ x = 1$.

a) Give boundary conditions at $x = 0$ and $x = 1$ to make this a well-posed problem.

b) Give a stable, convergent numerical approximation to this initial boundary value problem.

Justify your answers.

[7] Consider the equation

$$u_t = u_x + \epsilon u_{xx}$$

for $\epsilon > 0$, to be solved for $t > u$ $\ 0 \leq x \leq 1$ with periodic boundary conditions at $x = 0, 1$ and $u(x,0) = u_0(x)$ smooth and periodic of period 1.

a) Construct an explicit finite difference scheme which is stable and convergent in L^2 with a time step restriction of the type

$$\Delta t \leq a\Delta x + b\epsilon(\Delta x)^2$$

with $0 < a, b$ fixed and independent of ϵ and Δx. Justify your answer.

b) Show that this method is also stable and convergent in the maximum norm.