Qualifying Exam, Fall 2002

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

Do 5 problems and only 2 of them from 1, 2, and 3

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[1] Consider using the composite trapezoidal method to numerically evaluate the following integral

$$\text{(I)} \qquad \int_0^1 \frac{\sin(t)}{\sqrt{t}}\, dt$$

Two different methods are employed:

(i) The composite trapezoidal method is directly applied to the integral (I) and the value of the integrand at t=0 is taken to be 0.

(ii) The composite trapezoidal method is applied to

$$\text{(I}'\text{)} \qquad \int_0^1 2\sin(s^2)\, ds$$

(This latter integral is obtained from (I) by using the change of variables $s = \sqrt{t}$.)

The errors in the numerical approximation for these computations are giving in the following table

dx	Error With Computation (i)	Error With Computation (ii)
.02	$-5.840e-04$	$7.204e-05$
.01	$-2.068e-04$	$1.800e-05$
.005	$-7.325e-05$	$4.500e-06$

(a) What is the expected rate of convergence for the composite trapezoidal method?

(b) Give an estimate, based on the results in the above table, of the rate of convergence for each of the computational procedures.

(c) If your estimated rate of convergence does not agree with the expected rate of convergence for either of these procedures, explain this discrepancy.

[2] Consider the two point boundary value problem over the interval $[0, 1]$

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) = f(x) \qquad u(0) = u(1) = 0$$

with $p(x) > 0$.

(a) Assuming you are using an equispaced set of grid points in [0,1], give a finite difference discretization of this equation that results in a *symmetric* linear system of equations.

(b) Derive the leading term of the truncation error for the discretization in (a).

[3] Let A be an n by n non-singular matrix and consider iterative methods of the form

$$M\,\vec{x}^{n+1} = \vec{b} + N\,\vec{x}^{n}$$

where $A = M - N$.

(a) Assuming M is non-singular, state a sufficient condition that insures convergence of the iterates to the solution of $A\vec{x} = \vec{b}$ for any starting vector $\vec{x}^0$.

(b) Describe the matrices M and N for

(i) Jacobi's iteration
(ii) Gauss-Seidel iteration

(c) If A is strictly diagonally dominant, prove that Jacobi's method converges.

[4] Consider the two stage Runge-Kutta method

$$\begin{aligned} y^* &= y_n + a\Delta t f(y^n) \\ y^{n+1} &= y^n + \Delta t(b_1 f(y^n) + b_2 f(y^*)), \end{aligned}$$

for the ordinary differential equation $y' = f(y)$.

(a) Derive the equations for a, b_1, and b_2, that give a second order method.

(b) Find the interval of absolute stability for the method.

(c) Does the interval of absolute stability depend upon the coefficients a, b_1, b_2 ?

[5] Consider the second order equation

$$u_{tt} + 2bu_{tx} - a^2 u_{xx} - cu_x - du_t = 0$$

to be solved for $t > 0$, periodic in x, of period one.

(a) Write it as an equivalent first order system.

(b) For which values of the real numbers a, b is the corresponding initial value problem well posed?

(c) Set up a convergent finite difference approximation for the well posed initial value problem.

Justify your answers

[6] Consider the convection diffusion equation

$$u_t + au_x = u_{xx}$$

to be solved for $t > u$ $u(x, 0)$ given and $u(x, t)$ periodic in x, periodic, with the constant $a > 0$. Consider the difference approximation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{av_{i+1}^n - av_{i-1}^n}{2\Delta x} = \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2}$$

(a) For which values of $\Delta t, \Delta x, a$ do we have a scheme which satisfies a maximum principle as $\Delta x \to 0$? Are you happy with this result? Explain.

(b) Set up a scheme which is explicit, consistent and satisfies the maximum principal for

$$a\frac{\Delta t}{\Delta x} + 2\frac{\Delta t}{(\Delta x)^2} \leq 1$$

Explain.

[7] Consider the boundary value problem

$$-\Delta u + cu = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \gamma u = g \text{ on } \partial\Omega.$$

Here c and f are given smooth functions on Ω, and γ and g are given smooth functions on $\partial\Omega$.

(a) Give a variational formulation of the problem.

(b) Describe a piecewise linear Galerkin finite element approximation for the problem. Explain how the finite element method leads to a linear algebraic system.

(c) Under what conditions would you expect this to converge ?