Corrected version
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Qualifying Exam, Winter 2002

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

MA: Do any 5 problems.

Ph.D.: Do 5 problems and only 3 of them from 1, 2, 3, and 4.

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[1] Consider the equation

$$f(x) = x^2 - 2x + 1 \tag{1}$$

and let x^* be a solution of $f(x^*) = 0$.

(a) Give the iteration that results when Newton's method is employed to find the roots of (1).

(b) Let $e_n = x_n - x^*$ be the error at the nth step of this iteration. Derive a recurrence relation for the error that relates e_n to e_{n-1}.

(c) What is the rate of convergence of the iteration in (a)? Justify your answer.

[2] Consider the centered difference approximation to $f'(x)$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}.$$

(a) Derive an expression for the leading term of the truncation error of this approximation.

(b) Suppose that $f(x+\Delta x)$ and $f(x-\Delta x)$ have an evaluation error of size ε for all Δx, e.g.

$$f_c(x+\Delta x) = f(x+\Delta x) + \varepsilon_1$$

$$f_c(x-\Delta x) = f(x-\Delta x) + \varepsilon_2$$

where $f_c(x+\Delta x)$ and $f_c(x-\Delta x)$ are the "computed" values of f at $x+\Delta x$ and $x-\Delta x$ respectively with $|\varepsilon_1| < \varepsilon$ and $|\varepsilon_2| < \varepsilon$.
How does the error in the approximation of $f'(x)$ behave when the computed values are used and $\Delta x \to 0$?

(c) How should Δx be chosen in relation to ε so that the error in the approximation of $f'(x)$ is minimized when computed values are used?

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[3] Consider the following factorization of a tri-diagonal matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & * & * & * & \\ & & * & * & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ d_2 & 1 & & & \\ & * & * & & \\ & & * & * & \\ & & & d_n & 1 \end{pmatrix} \begin{pmatrix} e_1 & c_1 & & & \\ & e_2 & c_2 & & \\ & & * & * & \\ & & & * & c_{n-1} \\ & & & & e_n \end{pmatrix}$$

(a) Derive the recurrence relations that determine the values of the d_k's and e_k's in terms of the values of the a_k's, b_k's and c_k's.

(b) Give a condition on the matrix $\mathbf{A}$ which ensures your recurrence relations won't break down.

(c) Give the formulas that allow you to compute the solution of $\mathbf{A}\,\vec{x} = \vec{b}$ in $O(n)$ operations.

[4] Consider the second order Runge-Kutta method

$$\begin{aligned} y^* &= y^n + \Delta t\, \mathrm{F}(y^n) \\ y^{n+1} &= y^n + \frac{\Delta t}{2}\, \mathrm{F}(y^n) + \frac{\Delta t}{2}\, \mathrm{F}(y^*) \end{aligned}$$

for approximating solutions to the initial value problem

$$\frac{dy}{dt} = \mathrm{F}(y) \qquad y(t_0) = y_0.$$

Derive an error bound of the form

$$|e_n| \le C_1\, |e_0| + C_2\, (\Delta t)^2 \qquad n = 1, 2, \ldots N$$

where $e_n = y^n - y(t_n)$, $C_1$and C_2 are constants, and $\Delta t = \dfrac{(T - t_0)}{N}$. Please state your assumptions concerning F.

[**5**] For the equation

$$u_{tt} + u_t = u_{xx} + u_x$$

to be solved for $t > 0 \quad 0 < x < 1$ with periodic boundary conditions

$$u(0,t) = u(1,t),\ u_x(0,t) = u_x(1,t)$$

and initial data

$$\begin{aligned} u(x,0) &= \varphi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

(a) Restate this problem as an equivalent system of first order equations.

(b) Give a convergent second order accurate finite difference approximation to this first order system. Justify your answers.

[**6**] Consider constructing a numerical method to solve $u_t = u_{xx}$ for $t > 1 \quad 0, \leq x \leq 1$, with periodic boundary conditions:

$$u(0,t) \equiv u(1,t)$$

and smooth initial data

$$u(x,0) = \varphi(x)$$

Would you rather use the approximation (A) or (B):

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{A}$$

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^n - (u_i^{n+1} + u_i^{n-1}) + u_{i-1}^n}{(\Delta x)^2} \tag{B}$$

Describe the stability and convergence properties of both methods.

[7] Consider the Neumann problem

$$-(u_{xx} + u_{yy}) = f(x, y) \quad -1 \le x \le 1 \quad -1 \le y \le 1 \tag{A}$$

with

$$\frac{\partial u}{\partial \vec{n}} = g \tag{B}$$

($\vec{n}$ is the outwards unit normal) and the condition

$$\int_{|x|<1,\ |y|<1} u(x, y)dxdy = 0 \tag{C}$$

(a) Why do we need condition (C)?

Now replace (A) by

$$u - (u_{xx} + u_{yy}) = f \tag{A$'$}$$

and keep condition (B).

(b) Do we still need condition C? Why or why not?

(c) Set up a finite element method that converges for the problem (A′), (B).
Justify your answers.