

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

[1] (5 Pts.) For a single panel, Simpson's rule

$$\int_a^{a+2h} f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$$

is fifth order accurate.

(a) What is the order of accuracy of the composite Simpson's rule?

(b) I have written a program that implements composite Simpson's rule for integrating functions over the interval $[0, 1]$. In checking the program for correctness I test the routine on the integral $\int_0^1 x^5 dx$ and I obtain the following results

M	approximation	error
8	0.24169	0.04169
16	0.22083	0.02083
32	0.21041	0.01041
64	0.20520	0.00520

What is the factor by which the errors should decrease as the number of panels, M , is doubled?

(c) On the basis of the above computational results can I conclude that my program is incorrect? Explain your answer

[2] (5 Pts.) Consider the following iterative method

$$A \vec{x}^{k+1} = B \vec{x}^k + \vec{c}$$

where \vec{c} is the vector $(1, 1)^t$ and A and B are the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Assume the iteration converges; to what vector \vec{x} does the iteration converge?

(b) Does this iteration converge for arbitrary initial vectors, \vec{x}^0 ? Justify your answer.

Qualifying Exam, Fall 2003
NUMERICAL ANALYSIS

[3] (5 Pts.) (a) Give the cubic polynomial that interpolates the function $f(x) = 2^x$ at the points $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

(b) Give the value of your interpolant at $x = \frac{1}{2}$, and hence derive an approximation to $\sqrt{2} = 2^{\frac{1}{2}}$.

[4] (10 Pts.) (a) Construct a two-stage second order Runge-Kutta method for the ODE

$$y' = f(y), \quad y(0) = y_0,$$

and find its region of absolute stability.

(b) Give an equivalent first-order system for the second-order differential equation:

$$y'' - 21y' + 20 = 0.$$

(c) Give the stability time-step restriction if 2nd order Runge-Kutta is used to compute solutions to the first order system.

[5] (10 Pts.) Consider the differential equation

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)^2 \varphi - c^2 \frac{\partial^2 \varphi}{\partial x^2} = 0$$

to be solved for $0 \leq x \leq 1$, $t > 0$, with periodic boundary conditions in x and initial data

$$\begin{aligned} \varphi(x, 0) &= \varphi_0(x) \\ \varphi_t(x, 0) &= \varphi_1(x) \end{aligned}$$

Here $u, c > 0$ are positive constants. Give a convergent, second order accurate, finite difference approximation to this equation. Be sure to justify that your approximation is second order accurate and convergent.

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NUMERICAL ANALYSIS

[6] (10 Pts.) Consider the one-dimensional diffusion equations

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, \quad \alpha > 0$$

to be solved for $0 \leq x \leq 1$, $t > 0$, with periodic boundary conditions in x and initial data

$$v(x, 0) = v_0(x)$$

Assume one uses the Dufort Frankel method:

$$\frac{v_m^{n+1} - v_m^{n-1}}{2\Delta t} = \alpha \left(\frac{v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n}{\Delta x^2} \right)$$

as a means of computing approximate solutions to this equation.

(a) Determine the truncation error associated with this approximation. Under what conditions does the scheme provide a consistent approximation to the diffusion equation? Would the condition required for consistency be difficult to satisfy in a set of computational experiments where Δx is repeatedly halved?

(b) Surprisingly, this scheme is explicit and unconditionally stable. Show this, and explain why this does not violate the CFL condition.

[7] (10 Pts.) Develop and describe the piecewise-linear Galerkin finite element approximation of

$$\begin{aligned} -\Delta u + u &= f(x, y), & (x, y) \in T, \\ u &= g_1(x), & (x, y) \in T_1, \\ u &= g_2(y), & (x, y) \in T_2, \\ \frac{\partial u}{\partial n} &= h(x, y), & (x, y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x, y) \mid x > 0, y > 0, x + y < 1\} \\ T_1 &= \{(x, y) \mid y = 0, 0 < x < 1\} \\ T_2 &= \{(x, y) \mid x = 0, 0 < y < 1\} \\ T_3 &= \{(x, y) \mid x > 0, y > 0, x + y = 1\}. \end{aligned}$$

Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence.