Qualifying Exam, Winter 2003

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

Do 5 problems and only 2 of them from 1, 2, 3

[**1**] Let $f(0)$, $f(h)$ and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$ and $x = 2h$.

(**a**) Derive the coefficients c_0, c_1 and c_2 so that

$$Df_h(x) = c_0\, f(0) + c_1\, f(h) + c_2\, f(2h)$$

is as accurate an approximation to $f'(0)$ as possible.

(**b**) Derive the leading term of a truncation error estimate for the formula you derived in (**a**).

[**2**](**a**) Find and solve the normal equations used to determine the coefficients for a straight line that fits the following data in the least squares sense.

x_i	$f(x_i)$
-1	2
0	3
1	3
2	4

(**b**) Let A be an $m \times n$ matrix, with $m > n$ and the columns of A being independent. Given the QR factorization of A, i.e. A=QR, where Q's columns are orthonormal and R is upper triangular, what equations must you solve to find the least squares solution of the over-determined system of equations $A\vec{x} = \vec{b}$?

(**c**) Show that the Gram-Schmidt orthogonalization process applied to the columns of A leads to a QR factorization of the matrix A. (Specifically, give the elements of Q and R when the Gram-Schmidt process is written in matrix form.)

[3] Consider the scalar function $f : \mathbf{R} \to \mathbf{R}$. Let x^* be a root of f and x^n be an approximation to that root.

(a) Derive the formula for getting a "better" approximation to the root by setting x^{n+1} to be the root of the linear approximation to f obtained from the first two terms of the Taylor series approximation to f at x^n.

(b) What is the common name for the method you have derived?

(c) Consider $F : \mathbf{R}^n \to \mathbf{R}^n$. Using the approach in part **(a)**, derive a vector iteration for solving $F(\vec{x}) = 0$.

[4] Consider the theta method

$$y_{i+1} = y_i + h\Big[\theta f(t_i, y_i) + (1-\theta) f(t_{i+1}, y_{i+1})\Big]$$

to approximate the solution of the ordinary differential equation $y' = f(t, y)$.

(a) Find the order of the method as a function of the values of the parameter θ.

(b) Determine all values of θ such that the theta method is A-stable.

(c) What particular method is obtained for $\theta = 1$? Prove convergence of the method in this case $\theta = 1$ and state the necessary assumptions

[5] To solve

$$u_t + a u_x = 0 \text{ for } t > 0,\ 0 \le x \le 1$$

$u(x, 0) = \varphi(x)$ smooth, u periodic in x, $u(x+1, t) \equiv u(x, t)$ we use:

$$\frac{1}{2\Delta t}[(v_j^{n+1} + v_{j+1}^{n+1}) - (v_j^n + v_{j+1}^n)] + \frac{a}{2\Delta x}[v_{j+1}^{n+1} - v_j^{n+1} + v_{j+1}^n - v_j^n] = 0$$

For what values of $\frac{\Delta t}{\Delta x}$, if any, does this converge? At what rate? Explain your answers.

[6] Consider the differential equation

$$u_t = u_{xx} + u_{yy} + bu_{xy} \text{ for } t > 0, \quad 0 < x < 1, \quad 0 < y < 1$$

with $u = 0$ on the boundary, and $u(x, y, 0) = \varphi(x, y)$, a smooth function.

(a) For what values of b can you obtain a convergent, unconditionally stable finite difference scheme?

(b) Construct such a scheme. Explain your answers.

[7] (a) Develop and describe the piecewise linear Galerkin finite element approximation of

$$\begin{cases} -\Delta u + b(x)u &= f(x), \quad x = (x_1, x_2) \in \Omega \\ u &= 2, \quad x \in \partial\Omega_1 \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, \quad x \in \partial\Omega_2, \end{cases}$$

where

$$\begin{aligned} \Omega &= \{x | x_1 > 0, \ x_2 > 0, \ x_1 + x_2 < 1\} \\ \partial\Omega_1 &= \{x | x_1 = 0, \ 0 \le x_2 \le 1\} \cup \{x | x_2 = 0, \ 0 \le x_1 \le 1\} \\ \partial\Omega_2 &= \{x | x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\} \end{aligned}$$

and

$$0 < b \le b(x) \le B.$$

(b) Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence