Qualifying Exam, Fall 2004

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** Let $\bar{x}$ be a root of a continuously differentiable function f(x):$R \to R$.
If x* is an approximate root, then

(i) Derive an expression that relates the magnitude of the residual at x* to the magnitude of the error of the root x*.

(ii) Give an example of a function where the magnitude of the residual at x* over-estimates the error of the root x*.

(iii) Give an example of a function where the magnitude of the residual at x* under-estimates the error of the root x*.

[2] **(5 Pts.)** Consider the integration formula

$$\int_{-1}^{1} f(x)\,dx \approx f(\alpha_1)\,\beta + f(\alpha_2)\,\beta.$$

(i) Determine α_1, α_2, and β so that this formula is exact for all quadratic polynomials.

(ii) What is the minimal degree polynomial for which the formula with the coefficients derived in (i) is not exact?

(iii) What is the expected order of a composite integration method based upon the formula with coefficients derived in (i)?

[3] **(5 Pts.)** Let A $\in R^{nxm}$ and b $\in R^n$ with $m > n$. For $\sigma > 0$ consider the following minimization problem

$$\min_{x \in R^m} \left(\|Ax - b\|_2^2 + \sigma^2 \| x \|_2^2 \right)$$

Derive the equation that the optimal solution satisfies and explain why the optimal solution is unique.

[4] **(10 Pts.)** Show that the one-step method given by

$$
\begin{aligned}
k_1 &= f(t^n, y^n),\\
k_2 &= f\Big(t^n + \frac{h}{2}, y^n + \frac{h}{2}k_1\Big),\\
k_3 &= f(t^n + h, y^n + h(-k_1 + 2k_2))\\
y^{n+1} &= y^n + \frac{h}{6}[k_1 + 4k_2 + k_3]
\end{aligned}
$$

for solving $y' = f(t, y)$, is of third order.

[5] **(10 Pts.)** Given the second order partial differential equation

$$u_{tt} + 2bu_{tx} = a^2 u_{xx} + cu_x + du_t + eu + f(t, x)$$

to be solved for $t > 0$, $0 \le x \le 2\pi$, with $u(x, t)$ periodic in x of period 2π.

(a) For what values of a, b is the initial value problem with initial data

$$
\begin{aligned}
u(x, 0) &= u_0(x)\\
u_t(x, 0) &= u_1(x)
\end{aligned}
$$

well posed?

(b) Write a stable convergent finite difference approximation for this problem. Justify your answer.

Hint: you might consider making this into a first order system of equations.

[6] **(10 Pts.)** Consider the equation

$$u_t = u_{xx} + u_x$$

to be solved for $t > 0$, $0 \le x \le 2\pi$, with $u(x, t)$ periodic in x of period 2π, and initial data $u(x, 0) = u_0(x)$.

Write an unconditionally stable convergent second order accurate scheme for this method and prove that your scheme satisfies these properties.

[7] **(10 Pts.)** Let Ω be a sufficiently smooth and bounded domain in the plane and let the boundary Γ of Ω be divided into two parts Γ_1 and Γ_2. Give a variational formulation of the following problem:

$$\begin{aligned} -\triangle u + u = f & \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g & \qquad \text{on } \Gamma_1, \\ u = u_0 & \qquad \text{on } \Gamma_2, \end{aligned}$$

where f, u_0 and g are given functions satisfying some appropriate assumptions (that you should specify). Formulate a FEM for this problem, and discuss (verify) the assumptions of the Lax-Milgram Lemma.