Qualifying Exam, Winter 2004

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** Let $g(x)$ be a continuously differentiable function and consider the fixed point problem

$$x = g(x).$$

(a) What conditions on $g(x)$ and $\alpha, 0 < \alpha \leq 1$, will guarantee convergence of the iteration

$$\begin{aligned} x^* &= g(x_n) \\ x_{n+1} &= \alpha\, x^* + (1-\alpha)\, x_n \end{aligned}$$

to the solution $\bar{x}$ of the fixed point problem?

(b) Prove that under the conditions that you derived in **(a)** the solution $\bar{x}$ of the fixed point problem is unique.

[2] **(5 Pts.)** For a given value of $h > 0$ consider the two approximations to $f'(x)$

$$D_h\, f = \frac{f(x+h) - f(x)}{h} \qquad D_{2h}\, f = \frac{f(x+2h) - f(x)}{2h}$$

Derive the coefficients β_1 and β_2 so that the combination of approximations $\beta_1 D_h\, f + \beta_2 D_{2h}\, f$ is a second order approximation to $f'(x)$.

[3] **(5 Pts.)** Assume the points $\{x_i\}$, for $i = 1 \ldots n+1$, are distinct. Prove that the polynomial of degree $\leq n$ that interpolates the data $\{(x_i, f(x_i))\}$ is unique.

[4] **(10 Pts.)** Consider the linear two-step numerical method for solving $\frac{dy}{dt} = f(t,y)$,

$$y_{i+2} = y_{i+1} + dt\left[\frac{3}{2}f(t_{i+1}, y_{i+1}) - \frac{1}{2}f(t_i, y_i)\right].$$

(a) Is this method consistent? Explain.

(b) What is the order of this method? Show your work.

(c) Does this method converge? Explain.

(d) Find a necessary and sufficient condition for the linear stability of the method (show your analysis, but without solving explicitly the obtained set of inequalities in the complex domain).

[5] **(10 Pts.)** Consider the hyperbolic equation

$$u_t + u_x + 2u_y = 0$$

for $t > 0$, (x, y) in the square $[-1, 1] \times [-1, 1]$, and initial data

$$u(x, y, 0) = \varphi(x, y)$$

(a) Boundary conditions on u are imposed to be zero on which sides of the square? Why?

(b) Set up a finite difference approximation which converges to the correct solution. Justify your answer.

[6] **(10 Pts.)** Consider the equation

$$u_t = u_{xx}$$

to be solved for $t > 0$, $x \in [-1, 1]$, with periodic initial data

$$u(x, 0) = \varphi(x), \;\; \varphi(x + 2) \equiv \varphi(x)$$

and $u(x,t)$ periodic in x for $t > 0$. Give a fourth or higher order accurate convergent finite difference scheme. Justify your answer.

[7] **(10 Pts.)** Consider the following problem in a domain $\Omega \subset R^2$, with $\Gamma = \partial\Omega$:

$$\begin{aligned} -\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma, \end{aligned}$$

where β_1 and β_2 are constants.

(a) Choose an appropriate space of test functions V, and give a weak formulation of the problem.

(b) For any $v \in V$, show that

$$\int_\Omega \Big(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\Big) dx = 0.$$

(c) By analyzing the linear and bilinear forms, show that the weak formulation has a unique solution.

(d) Set up a convergent finite element approximation and discuss the linear system thus obtained.