DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** Let $\{x_n\}$ be a sequence such that $x_n \geq \bar{x} \; \forall \; n$ and $\lim_{n\to\infty} x_n = \bar{x}$. Assume $\exists$ constants α and $p > 0$ such that for sufficiently large n

$$x_{n+1} - \bar{x} \approx \alpha \, (x_n - \bar{x})^p$$

(a) Assuming $\bar{x}$ is known, give a derivation of a formula that estimates p in terms of $\bar{x}$ and some number of consecutive iterates of the sequence $\{x_n\}$.

(b) Assuming $\bar{x}$ is *unknown*, give a derivation of a formula that estimates p in terms of some number of consecutive iterates of the sequence $\{x_n\}$.

[2] **(5 Pts.)** Consider the forward and backward difference operators D^+ and D^- defined by

$$D^+f(x) = \frac{f(x+h) - f(x)}{h} \qquad D^-f(x) = \frac{f(x) - f(x-h)}{h}.$$

(a) Assuming f is smooth, derive asymptotic error expansions for each of these operators.

(b) What combination of $D^+f(x)$ and $D^+f(x)$ gives a second order accurate approximation to the derivative $f'(x)$? Justify your answer.

[3] **(5 Pts.)** Consider the following factorization of a tri-diagonal matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & * & * & * & \\ & & * & * & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ d_2 & 1 & & & \\ & * & * & & \\ & & * & * & \\ & & & d_n & 1 \end{pmatrix} \begin{pmatrix} e_1 & c_1 & & & \\ & e_2 & c_2 & & \\ & & * & * & \\ & & & * & c_{n-1} \\ & & & & e_n \end{pmatrix}$$

(a) Derive the recurrence relations that determine the values of the d_k's and e_k's in terms of the values of the a_k's, b_k's and c_k's.

(b) Give a condition on the matrix $\mathbf{A}$ which ensures your recurrence relations won't break down.

[4] **(10 Pts.)** **(a)** Find conditions on the coefficients a_1, a_2, p_1, p_2 so that the following Runge-Kutta method for $y' = f(t, y(t))$ is of order $m \geq 2$:

$$y_{n+1} = y_n + h\Big[a_1 f(t_n, y_n) + a_2 f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n))\Big].$$

(b) Show by an example that the order cannot exceed two.

(c) Analyze the linear stability of the scheme when $a_1 = 0$, $a_2 = 1$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$.

[5] **(10 Pts.)** Let $a(x, y)$ and $b(x, y)$ be smooth, positive, functions. Consider the equation

$$u_t = (a(x, y)u_x)_x + (b(x, y)u_y)_y$$

to be solved for $t > 0$, $(x, y) \in [0, 1] \times [0, 1]$, with smooth initial data $u(x, y, 0) = u_0(x, y)$ and periodic boundary conditions in x and y; $u(0, y, t) \equiv u(1, y, t)$, $u(x, 0, t) \equiv u(x, 1, t)$.

(a) Construct a second-order accurate, unconditionally stable, scheme for this equation. Justify the accuracy and stability properties of your scheme.

(b) Construct a second-order accurate, unconditionally stable, scheme for this equation that only requires the inversion of one dimensional operators. Justify the accuracy and stability properties of your scheme

[6] **(10 Pts.)** Consider the initial boundary value problem

$$u_t + a\, u_x = 0$$

where a is a real number, to be solved for $x \geq 0$ and $t \geq 0$, with smooth initial data $u(x, 0) = u_0(x)$.

(a) For a given value of the constant a, what boundary conditions, if any, are needed to solve this problem?

(b) Suppose the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2\lambda^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$, $j = 1, 2, \ldots$,and $n = 0, 1, 2, \ldots$ is used to approximate solutions to this equation.

Give stable boundary conditions for u_0^n. Justify your statements.

[7] **(10 Pts.)** The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\nabla \cdot \Big(a(\vec{x}) \nabla u(\vec{x}) \Big) &= f(\vec{x}), \ \vec{x} \in \Omega \subset R^2, \\ u(\vec{x}) &= u_0(\vec{x}), \ \vec{x} \in \Gamma_1, \\ \frac{\partial u(\vec{x})}{\partial x_1} + u(\vec{x}) &= 0, \ \vec{x} \in \Gamma_2, \\ \frac{\partial u(\vec{x})}{\partial x_2} &= 0, \ \vec{x} \in \Gamma_3, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1\}, \\ \Gamma_1 &= \{(x_1, x_2) : \ x_1 = 0, \ 0 \le x_2 \le 1\}, \\ \Gamma_2 &= \{(x_1, x_2) : \ x_1 = 1, \ 0 \le x_2 \le 1\}, \\ \Gamma_3 &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ x_2 = 0, \ 1\}, \end{aligned}$$

$$0 < A \le a(\vec{x}) \le B, \quad a.e. \text{ in } \Omega, \quad f \in L^2(\Omega),$$

and $u_0|_{\Gamma_1}$ is the trace of a function $u_0 \in H^1(\Omega)$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Set up a finite element approximation using P_1 elements, and a set of basis functions such that the associated linear system is sparse and of band structure. Discuss the linear system thus obtained, and give the rate of convergence.