Qualifying Exam, Fall 2006

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.
Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-3] and [4-7].

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[1] **(5 Pts.)** The following code, implementing the bisection method, was used to find the roots of $x^2 - 2x + 1 = 0$ using an initial interval of $a = -1.0$ and $b = 0.0$. The result was

Approximate root of x^2 - 2*x + 1 is -0.0000009537

(a) Is this program running correctly, and if not, what is the cause of the problem?

```
% bisect.m
fstring = 'x^2-2*x+1';          % target function
a = -1.0;                       % left starting endpoint
b = 0.0;                        % right starting endpoint
eps = 1.0e-06;                  % root error bound tolerance
c = (a+b)/2.0;                  % midpoint = approximate root
nMax = 100;                     % allow only nMax interations
n = 0;
while((abs(b-a) > 2.0*eps)&(n < nMax))
 eval(['x = a;',fstring,';']); fa = ans;   % evaluate the function at a
 eval(['x = c;',fstring,';']); fc = ans;   % evaluate the function at c
 if(fa*fc <= 0)                            % a root lies in the left interval
 b = c;
 else                                      % a root lies in the right interval
 a = c;
 end
 c = (a+b)/2.0;                            % midpoint = approximate root
 n = n+1;
end
sprintf(['Approximate root of ',fstring,' is %-15.10f \n'],c)
```

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[2] **(5 Pts.)** Gauss-Laguerre quadrature rules have the form

$$\int_0^\infty x^\alpha \, e^{-x} \, f(x) \, dx \simeq \sum_{j=1}^{N} w_j \, f(x_j)$$

where α is a constant.
(a) Consider using such formulas to create approximations to integrals of the form

$$\int_0^\infty g(x) \, dx.$$

Give the nodes and weights, $\tilde{x}_j$ and $\tilde{w}_j$, (derived from the nodes and weights of the Gauss-Lagurre rule) that give rise to approximations of the form

$$\int_0^\infty g(x) \, dx \simeq \sum_{j=1}^{N} \tilde{w}_j \, g(\tilde{x}_j)$$

(b) For what types of functions will the integration rules you derived in (a) be exact?

[3] **(5 Pts.)** Consider the task of approximating a function $f(x)$ by a linear combination of N functions $q_k(x)$, $k = 1 \ldots N$, e.g.

$$f(x) \simeq \sum_{k=1}^{N} c_k \, q_k(x) \qquad \text{for } x \in [0,1]$$

(a) Give the equations that determine the $c_k{}'s$ so that $\left\| f(x) - \sum_{k=1}^{N} c_k \, q_k(x) \right\|_2$ is minimized. (The norm is taken over the interval $[0,1]$.)

(b) How would you solve the resulting equations?

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[4] **(10 Pts.)** Consider the linear two-step method

$$y_{n+2} - 3y_{n+1} + 2y_n = h\Big[\frac{13}{12}f(t_{n+2}, y_{n+2}) - \frac{5}{3}f(t_{n+1}, y_{n+1}) - \frac{5}{12}f(t_n, y_n)\Big],$$

for solving $y'(t) = f(t, y(t))$, $y(t_0) = y_0$.

(a) Show that the order of the method is 2.

(b) Is this method convergent ?

(c) How would the numerical scheme perform when applied to the simple example $y'(t) = 0$, $y(0) = 1$ with the initial conditions $y_0 = 1$ and $y_1 \neq y_0$ obtained using a one-step method in the presence of roundoff errors ? Justify your answers.

[5] **(10 Pts.)** Consider the equation

$$u_{tt} = au_{xx} + 2bu_{xy} + cu_{yy}$$

to be solved for $t > 0,\ 0 \le x \le 1,\ 0 \le y \le 1$ with initial data:

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) \\ u_t(x, y, 0) &= u_1(x, y) \end{aligned}$$

and periodic boundary conditions

$$\begin{aligned} u(x + 1, y, t) &\equiv u(x, y, t) \\ u(x, y + 1, t) &\equiv u(x, y, t) \end{aligned}$$

(a) For what values of the constants a, b, c is this a well posed problem?

(b) Write a stable convergent finite difference scheme for this problem.

Justify your answers.

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[6] **(10 Pts.)** Consider the nonlinear equation

$$u_t + \left(\frac{u^2}{2}\right)_x = bu_{xx}$$

to be solved for $t > 0,\ 0 \le x \le 1$ with initial data

$$u(x,0) = u_0(x)$$

and periodic boundary conditions

$$u(x+1,t) \equiv u(x,t),$$

for $b > 0$, a positive constant

(a) Write a finite difference approximation to this problem that satisfies a maximum and minimum principle for all $b > 0$.

(b) As b goes to zero, what difficulties do you expect to see with solutions to the finite difference approximation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x} = \frac{b(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2} \quad ?$$

[7] **(10 Pts.)** Let Ω be an open, bounded and connected subset of R^2, with sufficiently smooth boundary. Consider the problem

$$-\frac{\partial}{\partial x}\Big((1+2x^2+3y^4)u_x\Big) - u_{yy} = f \text{ in } \Omega,$$

$$(1+2x^2+3y^4)u_x n_x + u_y n_y + \lambda u = g \text{ on } \Gamma = \partial\Omega,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\vec{n} = (n_x, n_y)$ is the outward unit normal to $\partial\Omega$, and $\lambda \ge 0$ is a constant.

(a) Give weak variational formulations of the problem, by considering the cases $\lambda = 0$ and $\lambda > 0$. Show that each of these formulations have one and only one solution (under additional conditions on u, f or g if necessary, that you will specify).

(b) In the case $\lambda > 0$, describe a FE approximation using P_1 elements, and a set of basis functions such that the corresponding linear system is sparse. In particular show that the corresponding finite dimensional problem has a unique solution.

(c) What would be a standard error estimate for (b) with P_1 elements function of the meshsize h ? (assuming convexity and sufficient regularity of Ω and of its boundary Γ).