Qualifying Exam, Winter 2006

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.
Do all 7 problems.
Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** Consider the fixed point problem $x = G(x)$ with a solution α. Assume that $G(x)$ is two times continuously differentiable and that $G'(\alpha) = 0$ but $G''(\alpha) = K \neq 0$.

(a) Show that if the initial iterate x^0 is sufficiently close to α then the fixed point iteration $x^{n+1} = G(x^n)$ converges to α quadratically.

(b) Give an estimate of the size of ϵ that insures the iteration $x^{n+1} = G(x^n)$ converges to α if $x^0 \in [\alpha - \epsilon, \alpha + \epsilon]$

[2] **(5 Pts.)** Let A be a square non-singular matrix and $\vec{x}$ be the solution to A$\vec{x}$ = b. Assume one has an approximate solution $\vec{z}$ with an associated residual $\vec{r} = \vec{b}$ - A$\vec{z}$. Give a derivation of the following relation between the norm of the error and the norm of the residual

$$\frac{||\,\vec{x} - \vec{z}\,||_2}{||\,\vec{x}\,||_2} \leq ||\,\mathrm{A}\,||_2\,||\,\mathrm{A}^{-1}\,||_2 \frac{||\,\vec{r}\,||_2}{||\,\vec{b}\,||_2}$$

[3] **(5 Pts.)** Given data points (x_i, y_i) for $i = 1 \ldots N+1$ with distinct ordinates, prove that the interpolating polynomial of degree at most N is unique.

[4] (10 Pts.) Consider the ordinary differential equation

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

(a) Give a derivation of the trapezoidal method in a manner analogous to the derivation of general linear multistep methods.

(b) Find the leading term of the local truncation error of the trapezoidal method. What is the global error of the method?

(c) Analyze the linear stability for the trapezoidal method and show that the method is A-stable.

[5] (10 Pts.) Consider the second order system of equations

$$u_{tt} = u_{xx} + u_{yy} + 2bu_{xy}$$

to be solved for $0 \leq x,\ y \leq 1$, periodic boundary conditions, and smooth initial data

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) \\ u_t(x, y, 0) &= u_1(x, y) \end{aligned}$$

(a) For which real values of b is this a well posed problem? Why?

(b) Set up a second order accurate, convergent finite difference scheme. Justify your answer.

[6] (10 Pts.) Consider the convection diffusion equation

$$u_t + au_x = bu_{xx}, \quad b > 0, \quad a \neq 0 \qquad \text{for} \qquad 0 \leq x \leq 1.$$

(a) Construct a second order accurate unconditionally stable scheme.

(b) Do you think it is uniformly stable in the maximum norm as $b \downarrow 0$? Justify your answers.

[7] (10 Pts.) Consider the problem

$$\begin{aligned} -\Delta u + u = f(x,y) \qquad & (x,y) \in \Omega, \\ u = 1 \qquad & (x,y) \in \partial\Omega_1, \\ \frac{\partial u}{\partial \vec{n}} + u = x \qquad & (x,y) \in \partial\Omega_2, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x,y) : \ x^2 + y^2 < 1\}, \\ \partial\Omega_1 &= \{(x,y) : \ x^2 + y^2 = 1, \ x \le 0\}, \\ \partial\Omega_2 &= \{(x,y) : \ x^2 + y^2 = 1, \ x > 0\}, \end{aligned}$$

and $f \in L^2(\Omega)$.

(a) Determine an appropriate weak variational formulation.

(b) Verify conditions on the corresponding linear and bilinear forms needed for existence and uniqueness of the solution.

(c) Assume that the boundary $\partial\Omega$ is approximated by a polygonal curve. Describe a finite element approximation using P_1 elements. Discuss the form and properties of the stiffness matrix and the existence and uniqueness of the solution of the linear system thus obtained. Give a rate of convergence.