Qualifying Exam, Spring 2007

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-3] and [4-7].

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[1] **(5 Pts.)** Consider the function $f(x) = \dfrac{x^2}{e^{x^2}-1}$. The results of evaluating this function with values of $x \to 0$ are shown in the following table:

x	$\dfrac{x^2}{e^{x^2}-1}$
10^{-2}	9.9500833332e-001
10^{-4}	9.9999950000e-001
10^{-6}	9.9999999966e-001
10^{-8}	9.9999696824e-001
10^{-10}	1.0248191152e+000
10^{-12}	1.#INF00000e+000

(a) Do these results indicate a problem with the function evaluation?

(b) If there is a problem, explain why the problem is occurring, and give pseudo-code for a programming strategy that would alleviate the problem.

[2] **(5 Pts.)** **(a)** Derive the nodes and weights of a product integration formula based on the one dimensional composite Trapezoidal method for functions defined over the region $[0,1]\times[0,1]$, e.g. derive the points (x_i, y_j) and values $w_{i,j}$ such that

$$\int\int_{[0,1]\times[0,1]} f(x,y)\,dx\,dy \approx \sum_i \sum_j f(x_i,y_j)\,w_{i,j}$$

(b) For smooth functions, the leading term of the truncation error for the composite Trapezoidal method with mesh size h is $O(h^2)$. What are the leading terms of the truncation error for the method you derived in **(a)**. Justify your answer.

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[3] **(5 Pts.)** Consider the system of ODE's

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

If $\vec{y}(t)$ and $\vec{z}(t)$ are any two solutions of this equation with distinct initial data then $\|\vec{y}(t) - \vec{z}(t)\| \to 0$ as $t \to \infty$. Let $\vec{y}^n$ and $\vec{z}^n$ be approximate solutions obtained with Euler's method starting with distinct initial data at time t_0, e.g. $\vec{y}^0 = \vec{y}(t_0)$ and $\vec{z}^0 = \vec{z}(t_0)$ with $\vec{y}(t_0) \neq \vec{z}(t_0)$. What timestep is required to insure that $\|\vec{y}^n - \vec{z}^n\| \to 0$ as $n \to \infty$? Justify your result.

[4] **(10 Pts.)** Consider the two-step method

$$y_{n+2} = y_n + 2\,\Delta t\, f(t_{n+1}, y_{n+1})$$

for solving $y' = f(t, y)$.

(a) Find the order of the method. Is this method convergent? (justify your answers).

(b) Show that the region of absolute stability for this method is the empty set.

[5] **(10 Pts.)** Consider the convection-diffusion equation

$$u_t + au_x = bu_{xx}, \quad a \neq 0,\ b > 0,\ a, b \text{ constant.}$$

to be solved for $t > 0,\ 0 \leq x \leq 1$ with $u(x,t)$ periodic in x and $u(x,0)$ given.

(a) Construct an explicit second order scheme of the form:

$$u_i^{n+1} = u_i^n + c_2 u_{i+2}^n + c_1 u_{i+1}^n + c_0 u_i^n + c_{-1} u_{i-1}^n + c_{-2} u_{i-2}^n$$

by using the Lax-Wendroff procedure (e.g. the procedure where one uses the Taylor series expansion

$$u(x, t + \Delta t) = u(x,t) + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt} + 0(\Delta t)^3$$

and replaces the t derivatives by x derivatives using the equation).

(b) Derive stability conditions involving Δt and Δx. Justify your statements.

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[6] **(10 Pts.)** Consider the equation

$$u_t = xu_x + yu_y$$

to be solved for $t > 0$ and $(x, y) \in [0,1] \times [0,1]$ with $u(x, y, 0)$ given.

(a) On what part of the boundary of $[0,1] \times [0,1]$ should u be specified? Why?

(b) Construct a stable, convergent, scheme approximating this problem. Justify your statements.

[7] **(10 Pts.)** Consider the *biharmonic problem* in a two-dimensional domain Ω with sufficiently smooth boundary,

$$\triangle\triangle u = f \text{ in } \Omega,$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma = \partial\Omega,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ.

(a) Show using a Green's formula that, for any $u \in H^2(\Omega)$ satisfying the above boundary conditions, we have

$$\int_\Omega |\triangle u|^2 dxdy = \int_\Omega \left\{(u_{xx})^2 + (u_{yy})^2 + (u_{xy})^2 + (u_{yx})^2\right\} dxdy.$$

(b) Give a weak variational formulation of the biharmonic problem and show that this has a unique solution u in an appropriate space of functions, that you will specify. Assume that $f \in L^2(\Omega)$. Justify your answers.

(c) Describe a finite element approximation of the problem using P_5 elements and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.

(d) Assume convexity and sufficient regularity of the domain Ω. State a standard error estimate for the approximation.