Qualifying Exam, Fall 2008

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

Problems 1-4 are worth 5 points; problems 5-8 are worth 10 points.

All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8].

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[1] (5 Pts.) To avoid computing e^x repeatedly we consider constructing an array of equispaced values of e^x and then, for a given value of x, use linear interpolation with the nearest array values to efficiently obtain an approximation for e^x.

(a) Give a derivation of the error formula for linear interpolation.

(b) Use your error formula to estimate the fewest number of equispaced values in $[0, 1]$ that are required to insure that the error in the approximate value for e^x is less than $\frac{1}{2} \times 10^{-6}$ for any $x \in [0, 1]$. Justify your estimate.

[2] (5 Pts.) Consider the linear second-order boundary value problem

$$\frac{d^2y}{dt^2} = q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta,$$

with q a continuous function satisfying $q(x) > 0$ on $[a, b]$.

(a) Formulate a finite-difference approximation to the problem using second order central difference approximations to the derivative. Choose the grid points $x_i = a + ih$, for $i = 0, ..., N + 1$, where $h = \dfrac{(b-a)}{N+1}$, $N > 0$ integer.

(b) Give the matrix/vector formulation that represents the $N \times N$ linear system of equations in (a), and, quoting a property on matrices, justify that this linear system has a unique solution.

[3] (5 Pts.) (a) Consider the linear system $A\vec{x} = \vec{b}$ with $\vec{x}, \vec{b} \in \mathbb{R}^n$ and $A = M - N \in \mathbb{R}^{n\times n}$ nonsingular. If M is nonsingular and if $(M^{-1}N)^k \to O$ as $k \to \infty$, show that the iterates $\vec{x}_k$, defined by

$$M\,\vec{x}_{k+1} = N\,\vec{x}_k + \vec{b},$$

converge to $\vec{x} = A^{-1}\vec{b}$ for any starting vector $\vec{x}_0$.

(b) Find a splitting $A = M - N$ for the matrix $A = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$, so that the iteration in (a) is convergent. Justify your answer.

[4] **(5 Pts.)** Consider the ordinary differential equation $\frac{dy}{dt} = f(t, y)$.

(a) Give a derivation of the Taylor series method that is of global second order accuracy.

(b) What is the interval of absolute stability for this method? Justify your answer.

[5] **(10 Pts.)** Consider the Trapezoidal method and Backward Euler method for the ordinary differential equation $\frac{dy}{dt} = f(y)$,

$$\text{(TM)} \quad y^{k+1} = y^k + \frac{dt}{2}\left(f(y^k) \; + \; f(y^{k+1})\right) \qquad \text{(BE)} \quad y^{k+1} = y^k + dt \; f(y^{k+1})$$

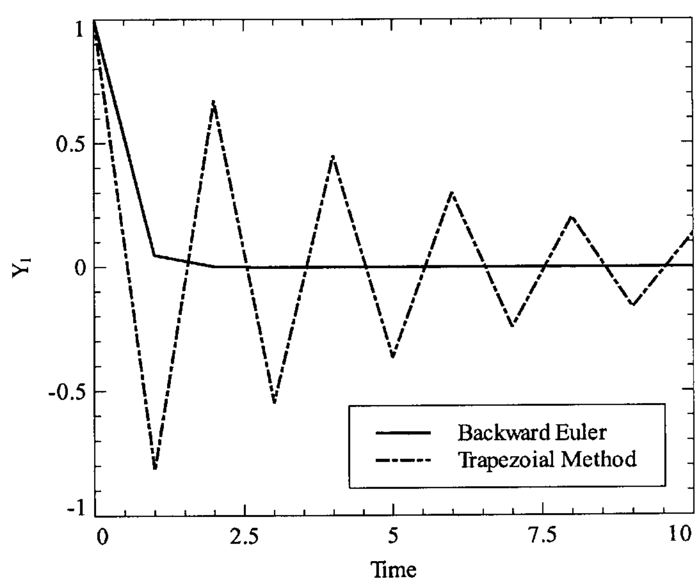
(a) Show that for each of these methods the interval $(-\infty, 0)$ is contained within its' interval of absolute stability.

(b) If these methods are applied to the following system of ordinary differential equations

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} -11 & -9 \\ -9 & -11 \end{pmatrix} \vec{y}, \qquad \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $dt = 1.0$, then, as shown in the plot below, the value of the first component of the numerical solution obtained with the Trapezoidal method exhibits undesirable oscillations while the first component of the numerical solution obtained with Backward Euler doesn't. Explain the presence of the oscillations in the solution obtained with the Trapezoidal method and the absence of oscillations obtained with Backward Euler. Justify your explanation.

(c) For what value of the timestep will the solution obtained with the Trapezoidal method cease to give an oscillatory solution? Justify your result.



[6] (10 Pts.) Consider the initial value problem

$$u_t = (a(x,y)u_x)_x + (b(x,y)u_y)_y$$

to be solved for $t > 0$, $0 \le x, y \le 1$, with periodic boundary conditions and

$$u(x,y,0) = u_0(x,y) \quad \text{given}$$

Here $a(x,y), b(x,y), u_0(x,y)$ are smooth $a(x,y), b(x,y) > \delta > 0$. Construct a second order accurate approximation which is unconditionally stable and which can be solved using tridiagonal inversions. Justify your answer.

[7] (10 Pts.) (a) Can you find a stable, convergent, numerical scheme to approximate solutions of

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0 \tag{1}$$

for $t > 0$, $0 \le x \le 1$, periodic boundary conditions, with smooth initial data? Justify your answer, and if such a scheme exists, prove your scheme is stable and convergent.

(b) Suppose we modify the equation to

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \epsilon \begin{pmatrix} u \\ v \end{pmatrix}_{xx}, \tag{2}$$

with the same initial and boundary conditions, for $\epsilon > 0$. Can you find a stable, convergent, numerical scheme? Justify your answer, and if such a scheme exists, prove your scheme is stable and convergent.

[8] (10 Pts.) Consider the problem of developing a piecewise-linear Galerkin finite element approximation of

$$\begin{aligned} -\Delta u + u &= f(x,y), \quad (x,y) \in T, \\ u &= g_1(x), \quad (x,y) \in T_1, \\ u &= g_2(y), \quad (x,y) \in T_2, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x,y) |\ x > 0,\ y > 0,\ x + y < 1\} \\ T_1 &= \{(x,y) |\ y = 0,\ 0 < x < 1\} \\ T_2 &= \{(x,y) |\ x = 0,\ 0 < y < 1\} \\ T_3 &= \{(x,y) |\ x > 0,\ y > 0,\ x + y = 1\}. \end{aligned}$$

(a) Determine an appropriate weak variational formulation.

(b) Verify conditions on the corresponding linear and bilinear forms needed for existence and uniqueness of the solution (specifying any necessary assumptions on the functions f, g_1, g_2, and h).

(c) Describe a finite element approximation using P_1 elements for this problem. Give the form and properties of the stiffness matrix that insure existence and uniqueness of solutions of the linear system thus obtained.