Qualifying Exam, Spring 2008

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-3] and [4-7].

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[1] (5 Pts.) Assume one is using a numerical procedure that results in a value V_h whose accuracy depends on the value of a parameter h. Let $\bar{V}$ be the "exact" value obtained in the limit as $h \to 0$ and assume that there is an asymptotic error expansion of the form

$$V_h - \bar{V} = c_1 h^2 + c_2 h^4 + c_3 h^6 + \cdots$$

(a) What values of α_k and h_k should be used so that the linear combination $\sum_{k=1}^{k=2} \alpha_k V_{h_k}$ is a 4th order approximation to $\bar{V}$? Show your work.

(b) Will the linear combination derived in (a) still converge to $\bar{V}$ in the limit as $h \to 0$ if the asymptotic error expansion has the form $c_1 h + c_2 h^2 + c_3 h^3 + \cdots$?

[2] (5 Pts.) Let A be an $n \times n$ singular symmetric matrix with eigenvalues λ_1=0 and $\lambda_i \neq 0$ for $i = 2, 3, \ldots n$. Under what conditions on the factor α, the vector $\vec{b}$, and the initial iterate, $\vec{x}^0$, will the iteration

$$\begin{aligned} \vec{x}^* &= \vec{b} - (\mathrm{A} - \mathrm{I})\, \vec{x}^n \\ \vec{x}^{n+1} &= \alpha\, \vec{x}^* + (1 - \alpha)\, \vec{x}^n \end{aligned}$$

converge to a solution of $\mathrm{A}\, \vec{x} = \vec{b}$? Justify your answer.

[3] (5 Pts.) Let $p(x)$ be the interpolating polynomial of degree n that interpolates the $n+1$ data points (x_i, y_i) for $i = 0 \ldots n$ where $x_i \neq x_j$ if $i \neq j$. Derive the formula that incorporates $p(x)$ and yields the polynomial of degree $n+1$ that interpolates the original data points (x_i, y_i) for $i = 0 \ldots n$ and the additional distinct data point (x_{n+1}, y_{n+1}).

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[4] **(10 Pts.)** Consider the θ-method applied to the ordinary differential equation $y' = f(t, y)$,

$$y_{n+1} = y_n + h\Big[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\Big],$$

$n = 0, 1, \dots$ and $\theta \in [0, 1]$.

(a) Find the order of the method as a function of θ.

(b) Show that this method is $A-$ *stable* for $\theta = \frac{1}{2}$ (the trapezoidal rule).

[5] **(10 Pts.)** Consider the equation

$$u_{tt} = b\, u_{xy} + u_{xx} + u_{yy}$$

where b is a real number to be solved for $t > 0,\ \ 0 \le x,\ \ y \le 1$ with smooth initial conditions

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) \\ u_t(x, y, 0) &= u_1(x, y) \end{aligned}$$

and periodic boundary conditions

$$\begin{aligned} u(x+1, y, t) &\equiv u(x, y, t) \\ u(x, y+1, t) &= u(x, y, t) \end{aligned}$$

(a) For which values of b is this a well posed problem?

(b) For those b's, give a stable convergent finite difference scheme.

Justify your answers

[6] **(10 Pts.)** Consider the problem

$$u_t = -u\, u_x + \epsilon\, u_{xx}$$

for $t > 0,\ 0 \le x \le 1,$ and the constant $\epsilon > 0$.

(a) Given smooth initial conditions,

$$u(x, 0) = u_0(x),$$

construct a second order accurate solution which converges for some time interval $0 \le t \le T,\ T > 0$.

(b) In general, why will this scheme have problems being convergent as ϵ goes to zero?

Justify your answers.

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[7] (10 Pts.) Let U be a Hilbert space with norm $\|\cdot\|_U$. Suppose that $a(\cdot,\cdot)$ is a symmetric bilinear form on $U \times U$ and L a linear form on U such that

(i) $a(\cdot,\cdot)$ is continuous: there is $\gamma > 0$ such that $|a(v,w)| \leq \gamma \|v\|_U \, \|w\|_U \;\; \forall \; v, \, w \in U$
(ii) $a(\cdot,\cdot)$ is coercive: there is $\alpha > 0$ such that $|a(v,v)| \geq \alpha \|v\|_U^2 \;\; \forall \; v \in U$
(iii) L is continuous: there is $\Lambda > 0$ such that $|L(v)| \leq \Lambda \|v\|_U \;\; \forall \; v \in U$.

Consider the following abstract problems:

$$\text{(M)} \quad \text{Find } u \in U \text{ such that } F(u) = \min_{v \in U} F(v),$$

where $F(v) = \frac{1}{2} a(v,v) - L(v)$.

$$\text{(V)} \quad \text{Find } u \in U \text{ such that } a(u,v) = L(v) \;\; \forall v \in U.$$

(a) Show that problems (M) and (V) are equivalent, i.e., $u \in U$ is a solution of (M) if and only if u is a solution of (V).

(b) If u is a solution to these two problems, show the stability estimate

$$\|u\|_U \leq \frac{\Lambda}{\alpha}.$$

(c) If u_1 and u_2 are two solutions of (V), show that $u_1 = u_2$.

(d) Let $u \in U$ be solution of (V) and $u_h \in U_h$ (a finite dimensional subspace of U) be such that $a(u_h, v) = L(v) \;\; \forall v \in U_h$. Show that

$$\|u - u_h\|_U \leq \frac{\gamma}{\alpha} \|u - v\|_U \;\; \forall v \in U_h.$$