Qualifying Exam, Fall 2009

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f. Describe the Bisection algorithm for generating p_n, and show the approximation error formula

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when } n \geq 1.$$

[2] (5 Pts.) Let $f \in C^4[x_0 - h, x_0 + h]$, with $h > 0$. Derive the second-order central finite difference approximation of $f''(x_0)$ and give the error formula.

[3] (5 Pts.) Consider the following matrix

$$\mathrm{A} = \begin{pmatrix} a & -b \\ -a & a \end{pmatrix}$$

with $a > b > 0$. Prove that Gauss-Seidel converges to the solution of

$$\mathrm{A}\,\vec{\mathrm{u}} = \vec{\mathrm{f}}$$

for any non-trivial initial guess and any right hand side $\vec{f}$.

[4] (5 Pts.) A program was written that implements the composite trapezoidal method to approximate integrals of the form $\int_0^1 f(x)\,dx$. This program was tested on monomials of increasing degrees using double precision arithmetic and the number of panels $N = 1000$. The results are as follows:

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$f(x)$	Computed Approximation
1	1.000500000000
x	0.500500000000
x^2	0.333833500000
x^3	0.250500250000
x^4	0.200500333333

Is this program working correctly? Explain your answer.

[5] **(10 Pts.)** (a) When the forward Euler method is applied to the model problem

$$\frac{dy}{dt} = \lambda y, \qquad y(0) = y_0$$

with $\text{Re}(\lambda) > 0$, then for any timestep $dt > 0$, the quantity $dt\lambda$ is **not** contained in forward Euler's region of absolute stability. Should forward Euler not be used for such cases? Explain.

(b) Give a derivation of the leading term of the local truncation error for the backward Euler method

$$y^{k+1} = y^k - dt\, f(y^{k+1})$$

used to create approximate solutions to the initial value problem $\dfrac{dy}{dt} = f(y)$ with $y(t_0) = y_0$ for $t \in [t_0, T]$.

(c) Assuming $f(y)$ is smooth and has a global Lipschitz constant K, give a derivation of an error bound for the backward Euler method. Specifically, derive a bound for $|e_N| = |y(T) - y^N|$ where $dt = (T - t_0)/N$ and y^N is the approximate solution at T obtained using N steps of backward Euler.

[6] **(10 Pts.)** Consider the initial-boundary value problem:

$$u_t = a(x,t)u_x$$

to be solved for $t > 0$, $0 \le x \le 1$, with $u(x, 0) = \phi(x)$, and both $\phi(x)$ and $a(x, t)$ smooth.

(a) What boundary conditions, if any, should we impose at $x = 0$ and at $x = 1$? Why ?

(b) Construct a convergent finite difference approximation Justify your answers.

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[7] **(10 Pts.)** Consider the equation:

$$u_t = uu_{xx}$$

to be solved for $t > 0$, $0 \le x \le 1$, $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$ for $t > 0$, $u(x,0) = \phi(x)$, $\phi(x)$ smooth.

(a) What restrictions do you need on $\phi(x)$ in order to be able to obtain a convergent finite difference scheme ? Why ?

(b) Given such a ϕ, construct such a scheme. Justify your answers.

[8] **(10 Pts.)** Consider the boundary-value problem

$$\begin{aligned} -\Delta u &= f(x,y), && (x,y) \in \Omega \\ u &= 1 && (x,y) \in \partial\Omega_1 \\ \frac{\partial u}{\partial n} + u &= x && (x,y) \in \partial\Omega_2, \end{aligned}$$

where

$\Omega = \{(x,y) |\ x^2 + y^2 < 1\}$,
$\partial\Omega_1 = \{(x,y) |\ x^2 + y^2 = 1,\ x \le 0\}$,
$\partial\Omega_2 = \{(x,y) |\ x^2 + y^2 = 1,\ x > 0\}$
and $f \in L^2(\Omega)$.

(a) Write a weak variational formulation by choosing the appropriate space of test functions.

(b) Verify the assumptions of the Lax-Milgram Theorem.

(c) Describe and analyze in details a piecewise-linear Galerkin finite element approximation for the problem.