Qualifying Exam, Spring 2009

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Give a derivation of the nodes and weights of a Gaussian quadrature formula $\sum_{i=1}^{n} c_i f(x_i)$ with $n = 2$ of highest order that approximates the integral $\int_{-1}^{1} f(x)dx$. What is the order of the approximation?

[2] (5 Pts.) Let $a > 0$ and consider using Newton's method to find $\sqrt{a}$ by finding the roots of $x^2 - a = 0$.

(a) Assume the initial iterate x_0 is chosen so $x_0 > 0$. Show that if Newton's method converges it does so to $\sqrt{a}$ and not $-\sqrt{a}$.

(b) Describe the behavior of the Newton iterates if you accidently used $x^2 + a = 0$ instead of $x^2 - a = 0$.

[3] (5 Pts.) Recall that an $n \times n$ matrix A is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j=1,\ j\neq i}^{n} |a_{i,j}|$$

holds for each $i = 1, 2, ..., n$. Show that the linear system $Ax = b$ has a unique solution if A is diagonally dominant.

[4] (5 Pts.) Let I_f denote the integral of a function $f(x)$ over an interval $[a, b]$ and let I_h denote a composite Simpson's rule approximation to this integral obtained with a mesh size h. The order of the leading term of the asymptotic error expansion for composite Simpson's rule is 4.

(a) Derive the combination of I_h and $\mathrm{I}_{\frac{h}{2}}$ that yields a good approximation to the *value* of the leading term of the error expansion associated with the approximation $\mathrm{I}_{\frac{h}{2}}$.

(b) Suppose you have implemented a composite Simpson's rule routine and are generating a sequence of approximations by decreasing the mesh size by a factor of 2 for each successive approximation. What stopping criterion would you use to ensure that the error in the approximation is less than some tolerance ϵ? Justify your result.

[5] **(10 Pts.)** The corrected Heun's method is given by

$$\begin{aligned} y^p &= y^k + dt\ f(y^k) \\ y^C &= y^k + \frac{dt}{2}\left(f(y^k)\ +\ f(y^p)\right) \\ y^{k+1} &= y^k + \frac{dt}{2}\left(f(y^k)\ +\ f(y^C)\right) \end{aligned}$$

while the standard Heun's method consists of using only the first two of the above equations and setting $y^{k+1} = y^C$.

Consider applying the corrected Heun's method to the model problem

$$\frac{dy}{dt} = \lambda y, \qquad y(0) = y_0$$

(a) Based on the difference equation that results, derive the leading term of the local truncation error associated with corrected Heun's method applied to this model problem. Justify your answer.

(b) How much smaller do you expect the error to be when using corrected Heun's method instead of the standard Heun's method? Explain.

(c) Recall that the interval of absolute stability for the standard Heun's method is $[-2, 0]$. Does the corrected Heun's method have a larger region of absolute stability? Justify your answer.

[6] **(10 Pts.)** Consider the two initial-boundary value problems

$$\text{P1: } u_t = \left(x - \frac{1}{2}\right) u_x, \qquad \text{P2: } u_t = -\left(x - \frac{1}{2}\right) u_x$$

to be solved for $0 \le x \le 1,\ \ t > 0$ with $u(x,0) = f(x)$.

(a) What boundary conditions do we need to impose for each of these problems at $x = 0$ and $x = 1$?

(b) Set up a stable, convergent method for each of the resulting initial-boundary value problems.

Justify your answers.

[7] **(10 Pts.)** Consider the constant coefficient convection diffusion equation

$$u_t + a u_x = b u_{xx} \qquad \text{with } |a|,\ b > 0$$

to be solved for $t > 0,\ 0 \le x \le 1$, with periodic boundary conditions and $u(x,0) = f(x)$.

Construct an unconditionally stable, convergent method that is at least second order accurate. Justify your answer.

[8] **(10 Pts.)** Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Let $\alpha > 0$, $b \in L^\infty(\Omega)$ with $b \geq 0$ a.e. in Ω, $f \in L^2(\Omega)$, and $a_{ij} \in L^\infty(\Omega)$ for all $i, j = 1, ..., n$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \text{ for all } x \in \Omega, \ \xi \in \mathbb{R}^n.$$

Let $\Gamma_0 \subset \Gamma$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$ with positive measures and consider the problem

$$\begin{aligned} -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}\partial_{x_i}u) + bu &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_0, \\ \sum_{i,j=1}^{n} a_{ij}(\partial_{x_i}u)n_j &= 0 \text{ on } \Gamma_1, \end{aligned}$$

where $\vec{n} = (n_1, ..., n_n)$ is the unit exterior normal along the boundary $\Gamma = \partial\Omega$.

(a) Give a weak variational formulation of the problem, and show that this formulation of the problem has a unique solution.

(b) Setup a convergent finite element formulation of the problem using P_1 elements Specifically, give the main properties of the linear system that must be solved; show that the linear system has a unique solution and give a rate of convergence. Justify your answers.