Qualifying Exam, Fall 2010

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Simpson's rule with its error term for numerical integration is given by

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] - \frac{h^5}{90} f^{(4)}(\xi),$$

where $f \in C^4[x_0, x_2]$ and $x_1 - x_0 = x_2 - x_1 = h > 0$. Assume that $f \in C^4[a, b]$, n even, $h = (b-a)/n$, and $x_j = a + jh$, $j = 0, 1, ..., n$.

Show that there exists $\mu \in (a, b)$ for which the composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{3}\Big[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\Big] - \frac{b-a}{180} h^4 f^{(4)}(\mu).$$

[2] **(5 Pts.)** Let $g : [a, b] \mapsto [a, b]$ be a continuously differentiable function (g' continuous). Assume that there is a constant $0 < k < 1$ such that $|g'(x)| \leq k$ for all $x \in (a, b)$. Let $p \in [a, b]$ be a unique fixed point of g. For any $p_0 \in [a, b]$, define the sequence $p_n = g(p_{n-1})$, $n \geq 1$.

(a) Show that the sequence p_n converges to p.

(b) If $g'(p) \neq 0$, show that p_n converges only linearly to p.

[3] **(5 Pts.)** Let x be the solution of $\mathrm{A}x = b$ and $\tilde{x}$ be the solution of $\mathrm{A}\tilde{x} = \tilde{b}$ where A is an N × N matrix.

(a) Define $\kappa_2(\mathrm{A})$ the condition number of A using the 2-norm.

(b) Give a derivation of the error bound

$$\frac{||x - \tilde{x}||_2}{||x||_2} \leq \kappa_2(\mathrm{A}) \frac{||b - \tilde{b}||_2}{||b||_2}$$

[4] **(5 Pts.)** Assume that all the roots of the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0$$

are real and none are repeated. Let ξ_n be largest root (in modulus) of $p(x)$. Show that under suitable restrictions on the n initial values $\{\beta_k\}_{k=0\ldots n-1}$, ξ_n can be determined by

$$\xi_n = \lim_{k\to\infty} \frac{z_{k+1}}{z_k}$$

where z_k is the solution to the nth order homogeneous linear difference equation

$$z_k = \beta_k \text{ for } k = 0, \ldots n-1$$

$$a_n z_k + a_{n-1} z_{k-1} + a_{n-2} z_{k-2} + \ldots + a_0 z_{k-n} = 0 \quad k = n,\, n+1, \ldots$$

[5] **(10 Pts.)** Let $f(p)$ be a real valued smooth function. For the two dimensional system of ODE's

$$\begin{aligned} \frac{dp}{dt} &= f(q) \\ \frac{dq}{dt} &= p \end{aligned}$$

consider the numerical method

$$\begin{aligned} p^n &= p^{n-1} + h\, f(q^{n-\frac{1}{2}}) \\ q^{n+\frac{1}{2}} &= q^{n-\frac{1}{2}} + h\, p^n \end{aligned}$$

with h the timestep, and $q^{\frac{1}{2}}$ obtained using $q^{\frac{1}{2}} = q^0 + \frac{h}{2} p^0$

(a) Derive the order of the local truncation error for this method. Show your work.

(b) For the linear case, when $f(q) = \alpha\, q$ where α is a real constant, give a derivation of an error bound for

$$\vec{e}^n = \begin{pmatrix} p^n \\ q^{n+\frac{1}{2}} \end{pmatrix}$$

in terms of a bound for the local truncation error and errors in the data $(p^0,\, q^{\frac{1}{2}})^t$.

(c) In the derivation of your error bound, is there a constraint on the timestep h that must be satisfied in order that your error bound hold?

[6] **(10 Pts.)** Consider the initial value problem

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

to be solved for $0 \leq x \leq 1, \quad t > 0$ with periodic boundary conditions

$$u(0,t) = u(1,t)$$
$$v(0,t) = v(1,t)$$

and initial data

$$u(x,0) = \phi(x)$$
$$v(x,0) = \psi(x)$$

(a) Construct a second order accurate convergent difference approximation for this.

(b) Suppose the initial data is

$$\varphi(x) = x \quad 0 \leq x \leq \tfrac{1}{2}$$
$$\varphi(x) = x - 1 \quad \tfrac{1}{2} \leq x \leq 1$$
$$\psi(x) \equiv 0$$

Where will the results actually be second order accurate in (x,t) space?

Justify your answers.

[7] **(10 Pts.)** Consider the initial value problem

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx}$$

for $\epsilon \geq 0$ to be solved for $0 \leq x \leq 1, \quad t > 0$ with periodic boundary conditions

$$u(0,t) = u(1,t)$$

and initial data $u(x,0) = \Phi(x)$

(a) Construct a second order accurate finite difference scheme which converges for all values of $\epsilon > 0$.

(b) Construct a scheme which converges for $\epsilon = 0$

Justify your answers.

[8] **(10 Pts.)** The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\mathrm{div}\Big(a(x)\nabla u(x)\Big) &= f(x), \ x \in \Omega, \\ u(x) &= 2, \ x \in \partial\Omega_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0, \ x \in \partial\Omega_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0, \ x \in \partial\Omega_3, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1\}, \\ \Gamma_1 = \partial\Omega_1 &= \{(x_1, x_2) : \ x_1 = 0, \ 0 \le x_2 \le 1\}, \\ \Gamma_2 = \partial\Omega_2 &= \{(x_1, x_2) : \ x_1 = 1, \ 0 \le x_2 \le 1\}, \\ \Gamma_3 = \partial\Omega_3 &= \{(x_1, x_2) : \ 0 < x_1 < 1, \ x_2 = 0, \ 1\}, \end{aligned}$$

and

$$0 < A \le a(x) \le B.$$

(a) Determine an appropriate weak formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness (assume $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$).

(c) Briefly describe a finite element approximation of the problem using P_1 elements, and a set of basis functions such that the linear system that you will obtain is sparse and of band structure.