Qualifying Exam, Spring 2010

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Let $S(x)$ be a cubic spline with knots $t_0, t_1, t_2, \dots t_n$. If it is determined that $S(x)$ is linear over $[t_1, t_2]$ and $[t_3, t_4]$ what can be said about $S(x)$ over $[t_2, t_3]$?

[2] **(5 Pts.)** Consider the iteration

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f(x_n)$$

for finding the roots of a two times continuously differentiable function $f(x)$. Assuming the method converges to a simple root x^*, what is the rate of convergence? Justify your answer.

[3] **(5 Pts.)** Let $P_{0,1,\dots,n} := P_{x_0,x_1,\dots,x_n}$ be the interpolating Lagrange polynomial of degree at most n through the points $x_0, x_1, \dots, x_n$ and values $f(x_0), \dots, f(x_n)$, such that $P_{0,1,\dots,n}(x_i) = f(x_i)$.

(a) Let $i, j \in \{0, 1, \dots, n\}$ be two distinct integers. Express $P_{0,1,\dots,n}$ in terms of $P_{0,\dots,i-1,i+1,\dots,n}$ and $P_{0,\dots,j-1,j+1,\dots,n}$.

(b) Suppose $x_j = j$ for $j = 0, 1, 2, 3$ and it is known that $P_{0,1}(x) = x + 1$, $P_{1,2}(x) = 3x - 1$, and $P_{1,2,3}(1.5) = 4$. Find $P_{0,1,2,3}(1.5)$.

[4] **(5 Pts.)** The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value of 4, and Simpson's rule gives the value 2. What is $f(1)$?

[5] **(10 Pts.)** Consider the numerical method

$$\vec{y}^{\,n+1} = \vec{y}^{\,n} + k\mathrm{A}\vec{y}^{\,n+1}$$

used to create approximate solutions of the linear system of equations

$$\frac{d\vec{y}}{dt} = \mathrm{A}\vec{y}, \qquad \vec{y}(t_0) = \vec{y}_0$$

for $t \in [t_0, \mathrm{T}]$.

(a) Derive a bound for the local truncation error in the $||*||_2$ norm of the form $\mathrm{C(T)}k^p$ where the constant C(T) is explicitly expressed in terms of $\sup_{t\in[t_0,\mathrm{T}]} ||\vec{y}\,''(t)||_2$ and powers of $||\mathrm{A}||_2$ and holds for $t \in [t_0, \mathrm{T}]$.

(b) Assume A is symmetric and negative definite. If $\vec{e}^{\,n} = \vec{y}^{\,n} - \vec{y}(t^n)$ is the error at the nth step and $\mathrm{C(T)}k^p$ the bound derived in **(a)**, show that

$$||\vec{e}^{\,n}||_2 \leq [\mathrm{T} - t_0]\,\mathrm{C(T)}k^{p-1}$$

for all n, $t_0 \leq nk \leq T$ assuming $\vec{e}^{\,0} = 0$.

Note: The defining equation for the local truncation error assumed for this problem is not based on the numerical method by k.

[6] **(10 Pts.)** Consider the equation

$$u_t = u_{xx} - c(x)u$$

with $c(x)$ smooth and positive, to be solved for $0 \leq x \leq 1$ and $t > 0$.

$$\begin{aligned} u(x,0) &= \varphi(x) \\ u(0,t) &= u(1,t) = 0 \end{aligned}$$

(a) Assuming $\varphi(x)$ is smooth, construct a finite difference scheme which converges in the maximum norm to the true solution. Justify your answers.

[7] (10 Pts.) Consider the initial value problem

$$u_{tt} = au_{xx} + 2bu_{xy} + cu_{yy}$$

to be solved for

$$\begin{aligned} 0 \le x, y &\le 1 \\ t &> 0 \end{aligned}$$

with

$$\begin{aligned} u(x,y,0) &= \varphi(x,y) \\ u_t(x,y,0) &= \psi(x,y) \end{aligned}$$

φ, ψ smooth and periodic with period 1 in x and y. Here a, b, c are real constants and u is periodic with period 1 in x and y.

(a) For which values of a, b, c is this well-posed?

(b) Set up a convergent finite difference scheme in the well-posed case.

Justify your answers.

[8] (10 Pts.) Consider the problem,

$$\begin{aligned} -\text{div}\Big(a(x)\nabla u\Big) + b(x)u = f(x), &\quad x = (x_1, x_2) \in \Omega, \\ u = 0, &\quad x \in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 2, &\quad x \in \partial\Omega_2, \end{aligned}$$

where $\Omega = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}$,
$\partial\Omega_1 = \{x|\ x_1 = 0,\ 0 \le x_2 \le 1\} \cup \{x|\ x_2 = 0,\ 0 \le x_1 \le 1\}$,
$\partial\Omega_2 = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$,
$0 < a \le a(x) \le A$, $0 < b \le b(x) \le B$, with a and b smooth functions and $f \in L^2(\Omega)$.

(a) Find the weak variational formulation and show that the problem is well-posed, by verifying the assumptions of the Lax-Milgram Lemma and by analyzing the appropriate bilinear and linear forms.

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.