Qualifying Exam, Fall 2011

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Assume $y(x)$ be a smooth function, and let $y(ih) = y_i$, $\frac{dy}{dx}(ih) = y'_i$ where h is the mesh width and i the grid point index. Determine the constants α and β so that the finite difference approximation

$$\alpha \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \beta \frac{y'_{i+1} - y'_{i-1}}{2h}$$

is a fourth order approximation to $\frac{d^2y}{dx^2}$.

[2] **(5 Pts.)** Consider using a finite difference method to create approximations to the following two-point boundary value problem with Neumann boundary conditions

$$\frac{d^2y}{dx^2} = f \quad \text{for } x \in (0,1) \quad \text{and} \quad \frac{dy}{dx}(0) = \frac{dy}{dx}(1) = 0$$

(a) Give the symmetrized system of linear equations that arises when a uniform mesh with N panels $(h = \frac{1}{N})$, is used and the second derivative is approximated by a three-point second order finite difference approximation and the equations are "closed" using a second order centered difference approximation to $\frac{dy}{dx}$.

(b) Is the linear system of equations in **(a)** singular? If so, state conditions on the discretization of f (e.g. the values f_i) that will insure that the linear system has a solution.

[3] **(5 Pts.)** Consider the initial-value problem

$$y' = x - x^2, \quad y(0) = 0.$$

Suppose we use Euler's method with stepsize h to compute approximate values y_i for $y(x_i)$, $x_i = ih$.

(a) Find an explicit formula for y_i and for $e(x_i, h) = y_i - y(x_i)$.

(b) Show that $e(x, h)$, for x fixed, goes to zero as $h \to 0$.

[4] **(5 Pts.)** Consider the implicit Euler's method (or the backwards Euler's method)

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

for the ODE $y' = f(x, y)$, with $y(0)$ the initial condition. Derive the region of absolute stability for the method. Given an ODE for which $\frac{\partial f}{\partial y} > 0$, does backwards Euler always give the qualitatively correct solution ? Explain.

[5] **(10 Pts.)** Consider the multistep method

$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = \frac{2h}{3} f_n$$

(a) Derive the leading term of the expansion of the local truncation error.

(b) Is this multistep method convergent? Explain.

(c) Consider applying this method to a system of equations of the form

$$\frac{d\vec{y}}{dt} = \mathbf{A}\vec{y}$$

where A is an $m \times m$ constant matrix. Give the implicit equations that must be solved at each timestep to advance the solution.

(d) Give the iteration that would result if Newton's method were used to solve the implicit equations.

(e) How many Newton iterations will be required to obtain the solution to the implicit equations?

[6] **(10 Pts.)** Consider the equation

$$u_t = b_1 u_{xx} + b_2 u_{yy}$$

$b_1, b_2 > 0$, to be solved in the unit square $0 \le x, y \le 1$, with $t > 0$. Assume the initial data

$$u(x, y, 0) = \phi(x, y)$$

is smooth, and the solution is to be periodic in x and y separately, with period 1.

(a) Devise a second order accurate convergent unconditionally stable scheme which involves only one dimensional matrix inversions.

(b) Justify your answer for **(a)**.

(c) Suppose we change the equation to

$$u_t = b_1 u_{xx} + b_2 u_{yy} + cu, \quad c > 0$$

with the same initial and periodicity conditions. Give a convergent second order accurate unconditionally stable scheme with the same simplicity as that in **(a)**.

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[7](**10 Pts.**) (**a**) Set up a well posed initial boundary value problem for the wave equation

$$u_{tt} = c^2(x,t)u_{xx}, \quad c(x,t) > 0$$

to be solved for $0 \leq x \leq 1, \quad t > 0$

(**b**) Devise a convergent finite difference approximation for this.

(**c**) Justify your answers.

[8] (**10 Pts.**) Give a variational formulation of the problem

$$\frac{d^4u}{dx^4} = f \quad \text{for } 0 < x < 1,$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied (assume that $f \in L^2(0,1)$). Which boundary conditions are essential and which are natural ? Develop and describe a finite element approximation of the problem using piecewise-cubic functions and uniform partition; describe the basis functions, the degrees of freedom of the finite-dimensional space and the corresponding linear system. Show that the linear system is sparse and has a unique solution.