Qualifying Exam, Spring 2011

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Give a derivation of the 5 point, 4th order, finite difference approximation to $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ using Richardson extrapolation. Assume a uniform grid spacing of size h.

[2] **(5 Pts.)** Let x^* be an approximate solution to the problem $f(x) = 0$ where f is continuously differentiable.

(a) Derive the approximate (first order) relation between the error in this approximation and the residual associated with this approximation.

(b) Give an example of a root finding problem and an approximate solution x^* for which the size of the residual associated with x^* is an *over* estimate of the size of the error associated with x^*. (A descriptive sketch of such a problem is acceptable).

(c) Give an example of a root finding problem and an approximate solution for which the size of the residual associated with x^* is an *under* estimate of the size of the error associated with x^*. (A descriptive sketch of such a problem is acceptable).

[3] **(5 Pts.)** Derive the Trapezoidal Rule with error term for numerical integration,

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(x_0) + f(x_1)\Big] - \frac{h^3}{12}f''(\xi),$$

for some $\xi \in (x_0, x_1)$, where $h = b - a$, $x_0 = a$, $x_1 = b$, and $f \in C^2[a, b]$.

[4] **(5 Pts.)** **(a)** Derive Newton's method for approximating a zero $p \in [a, b]$ of the function $f \in C^2[a, b]$, starting with an initial approximation $p_0 \in [a, b]$.

(b) Consider $f(x) = e^x - x - 1$. Do we expect any difficulty when applying Newton's method to approximate the zero $p = 0$ of f ? If yes, how could this be avoided ? How could we improve the rate of convergence ?

[5] **(10 Pts.)** Consider the numerical method

$$\begin{aligned}\vec{y}^{\,*} &= \vec{y}^{\,n} + k\mathrm{A}\vec{y}^{\,n+1}\\ \vec{y}^{\,n+1} &= \vec{y}^{\,n} + \frac{k}{2}\mathrm{A}\vec{y}^{\,n} + \frac{k}{2}\mathrm{A}\vec{y}^{\,*}\end{aligned}$$

used to create approximate solutions of the linear system of equations

$$\frac{d\vec{y}}{dt} = \mathrm{A}\vec{y}, \qquad \vec{y}(t_0) = \vec{y}_0$$

for $t \in [t_0, \mathrm{T}]$ and A is an $M \times M$ constant matrix.

(a) Derive the leading term of the local truncation error for this method.

(b) Assume that inexact computational arithmetic leads to a computed solution that doesn't satisfy the equations defining the method exactly, but instead satisfies

$$\begin{aligned}\vec{y}^{\,*} &= \vec{y}^{\,n} + k\mathrm{A}\vec{y}^{\,n+1} + \xi^{*}\\ \vec{y}^{\,n+1} &= \vec{y}^{\,n} + \frac{k}{2}\mathrm{A}\vec{y}^{\,n} + \frac{k}{2}\mathrm{A}\vec{y}^{\,*} + \xi^{n}\end{aligned}$$

Give a derivation of an error bound for this computed solution in terms of a bound on the error in the initial condition, a bound on the local truncation error, and a bound for the errors due to inexact arithmetic (ξ^{n}'s and ξ^{*}'s).

[6] **(10 Pts.)** Consider the differential equation

$$u_{tt} = \frac{\partial}{\partial x}(c(x,y)u_x) + \frac{\partial}{\partial y}(d(x,y)u_y)$$

to be solved for $t > 0 \;\; 0 \le x, y \le 1$, with initial conditions:

$$\begin{aligned}u(x,y,0) &= \Phi(x,y)\\ u_t(x,y,0) &= \psi(x,y)\end{aligned}$$

and periodic boundary conditions:

$$u(x+1,y,t) \equiv u(x,y,t), u(x,y+1,t) \equiv u(x,y,t),$$

with c, d, Φ, ψ smooth and $c, d > 0$
(a) Construct a convergent finite difference scheme which is second order accurate.
(b) Justify your statement

[7] (10 Pts.) Consider the equation

$$u_t + uu_x = \epsilon u_{xx},$$

for $\epsilon > 0$ constant, to be solved for $t \geq 0$, $u(x,0) = \Phi(x)$ and $0 \leq x \leq 1$ with periodic boundary conditions:

$$u(x+1,t) \equiv u(x,t)$$

(a) Construct a finite difference scheme which converges with a rate independent of ϵ.

(b) Justify your statement.

[8] (10 Pts.) Consider the problem,

$$\begin{aligned} -\Delta u + u &= f(x), \quad && x = (x_1, x_2) \in \Omega, \\ u &= 0, && x \in \partial\Omega_1, \\ \frac{\partial u}{\partial \vec{n}} + u &= x, && x \in \partial\Omega_2, \end{aligned}$$

where $\Omega = \{x = (x_1, x_2) |\ x_1^2 + x_2^2 < 1\}$, $\vec{n}$ is the exterior unit normal at $\partial\Omega$,
$\partial\Omega_1 = \{x = (x_1, x_2) |\ x_1^2 + x_2^2 = 1,\ x_1 \leq 0\}$,
$\partial\Omega_2 = \{x = (x_1, x_2) |\ x_1^2 + x_2^2 = 1,\ x_1 > 0\}$,
and $f \in L^2(\Omega)$.

(a) Derive the weak variational formulation of the problem and analyze the assumptions of the Lax-Milgram Lemma.

(b) Develop and describe a piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.