Qualifying Exam, Fall 2012

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Assume that $f(x) : \mathbb{R} \to \mathbb{R}$ is a smooth function. Let $\epsilon > 0$ and consider the three data values $(0, f(0))$, $(\epsilon, f(\epsilon))$, and $(1, f(1))$. Let $p(x)$ be the polynomial that arises as the limit of the polynomial interpolant of the data as $\epsilon \to 0$.

(a) What is the degree of $p(x)$?

(b) What data (if any) does $p(x)$ interpolate?

(c) What data (if any) does $p'(x)$ interpolate?

[2] **(5 Pts.)** Assume that $f(x) : \mathbb{R} \to \mathbb{R}$ is a smooth function with a simple root at $x = x^*$. Let x^k and x^{k+1} be two successive approximate roots close to x^* obtained using Newton's method. Explain why $|x^{k+1} - x^k|$ is a good approximation to the error $|x^k - x^*|$.

[3] **(5 Pts.)** Find a bound for the number of iterations of the Bisection method needed to achieve an approximation with accuray 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Justify your answer.

[4] **(5 Pts.)** For a single panel, the Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3}f''(\xi)$$

(where $x_1 - x_0 = x_0 - x_{-1} = h$, $x_{-1} < \xi < x_1$) is third order accurate.

What is the order of accuracy of the composite Midpoint rule? Justify your answer.

[5] (10 Pts.) Consider the numerical method

$$y^* = y_{n-1} + \frac{2\,h}{3}\,f(y_{n-1})$$

$$y_n = y_{n-1} + \frac{h}{4}\,f(y_{n-1}) + \frac{3\,h}{4}\,f(y^*)$$

to obtain approximate solutions to

$$\frac{dy}{dt} = f(y) \qquad y(t_0) = y_0$$

(a) Assuming $f(y) : \mathbb{R} \to \mathbb{R}$ is smooth, give the leading term of the expansion of the local truncation error for this method.

(b) Derive the relation between $|e_n| = |y(t_n) - y_n|$ and $|e_{n-1}| = |y(t_{n-1}) - y_{n-1}|$ and the local truncation error. You may assume that $f(y)$ has global Lipschitz constant L.

[c] Give the derivation of an error bound that uses your results from [**a**] and [**b**] to obtain an error bound for this method over a time interval $[0, \mathrm{T}]$.

[6] (10 Pts.) Consider the initial value problem

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= v_x \end{aligned}$$

to be solved for $0 \le x \le 1$, $t \ge 0$ with initial and boundary conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \qquad & u(1,t) &= u(0,t) \\ v(x,0) &= \psi(x), \qquad & v(1,t) &= v(0,t) \end{aligned}$$

where $\varphi(x)$ and $\psi(x)$ are smooth and periodic functions.

[6a](i) Can you write a stable, convergent finite difference scheme for this problem?

[6a](ii) Give an example if one exists.

Explain your answers.

[6b] Consider the related system

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= \frac{1}{1000} u_x + v_x \end{aligned}$$

with the same initial and boundary conditions.

[6b](i) Can you write a stable, convergent finite difference scheme for this problem?

[6b](ii) Give an example if one exists.

Explain your answers.

[7]**(10 Pts.)** Consider the differential equation

$$u_t = u_{xx} + u_{yy} + cu, \quad c < 0$$

with smooth initial data

$$u(x, y, 0) = u_0(x, y)$$

with $u_0(x, y)$ and $u(x, y, t)$ periodic, period 1 in x and in y.

(a) Show the solution decays in time for any initial data.

(b) Construct a stable, convergent finite difference scheme which is second order accurate in space and time and whose solutions have a similar decay in time.

(c) Justify your answer.

[8] **(10 Pts.)** Consider the boundary-value problem

$$\begin{aligned} -\triangle u &= f(x, y), && (x, y) \in \Omega \\ u &= 1 && (x, y) \in \partial\Omega_1 \\ \frac{\partial u}{\partial n} + u &= x && (x, y) \in \partial\Omega_2, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x, y) |\ x^2 + y^2 < 1\}, \\ \partial\Omega_1 &= \{(x, y) |\ x^2 + y^2 = 1,\ x \le 0\}, \\ \partial\Omega_2 &= \{(x, y) |\ x^2 + y^2 = 1,\ x > 0\} \end{aligned}$$

and $f \in L^2(\Omega)$.

(a) Write a weak variational formulation by choosing the appropriate space of test functions.

(b) Verify the assumptions of the Lax-Milgram Theorem.

(c) Describe and analyze in detail a piecewise-linear Galerkin finite element approximation for the problem.