Qualifying Exam, Spring 2012

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] **(5 Pts.)** Let A be an $m \times n$ matrix and b $\in R^n$ with $m > n$.

(a) Assume that the rank(A) is n. Derive the equations that determine the solution to

$$\min_{x \in R^m} \left(\|\mathrm{A}x - b\|_2^2 \right)$$

(b) Outline a procedure for obtaining the solution to these equations that avoids problems due to ill-conditioning that may occur if one uses Gaussian elimination on the equations directly.

[2] **(5 Pts.)** Consider the polynomial $p(x)$

$$p(x) = 1 + x + x(x-1) + \frac{1}{6}x(x-1)(x-2) + \frac{1}{24}x(x-1)(x-2)(x-3) + q(x)$$

(a) Determine a fifth degree polynomial $q(x)$ so that $p(x)$ interpolates the data $\{$(-1, -3), (0, 1), (1, 2), (2, 5), (3, 11), (4, 22)$\}$.

(b) Is $q(x)$ unique?

[3] **(5 Pts.)** Determine constants c_1, c_2, x_1 and x_2 such that the integration formula

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

has degree of precision 3.

[4] **(5 Pts.)** Consider the matrix

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

(a) Derive the Jacobi iteration for solving the linear system $A\,x = b$, for some vector $b \in \mathbb{R}^3$.

(b) Is this iteration convergent ? Justify your answer.

[5] (10 Pts.) Consider the following two methods for advancing the solution of $\frac{dy}{dt} = \mathrm{F}(y)$ with timestep k,

$$\text{(A)} \quad y_{n+1} = y_n + k\,\mathrm{F}(y_n)$$

$$\text{(B)} \quad y_{n+1} = y_n + \frac{k}{2}\,\mathrm{F}(y_n) + \frac{k}{2}\,\mathrm{F}(y_n + \frac{k}{2}\,\mathrm{F}(y_n))$$

(a) Derive the leading term of the expansion of the local truncation error for each method.

(b) Assuming that y_n is the exact solution at time t_n (e.g. $y_n = y(t_n)$) derive the formula that combines the values computed with each method, y^{A}_{n+1} and y^{B}_{n+1}, to provide an estimate of the error of method (B), $e_{n+1} = y^{\mathrm{B}}_{n+1} - y(t_{n+1})$.

(c) Assuming that y_n is the exact solution at time t_n (e.g. $y_n = y(t_n)$) and that y^{A}_{n+1} and y^{B}_{n+1} have been evaluated, derive the formula for the timestep k^* so that the approximation solution recomputed with method (B) using k^* has an estimated error at time t_{n+1} of magnitude less than ϵ.

Show your work.

[6] (10 Pts.) Consider the initial boundary value problem

$$u_{tt} = u_{xx}$$

to be solved for $t > 0 \quad 0 \le x \le 1$ with initial conditions

$$\begin{aligned} u(x,0) &= \Phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

with $\Phi(x), \psi(x)$ smooth and vanishing near $x = 0$ and $x = 1$. Take boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

(a) Write a stable, convergent, finite difference scheme for this problem.

(b) Justify your answer.

[7](10 Pts.) Consider the initial value problem

$$u_t = \frac{\partial}{\partial x}(a(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(b(x,y)\frac{\partial u}{\partial y})$$

to be solved for $t > 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$ with initial data

$$u(x,y,0) = \Phi(x,y)$$

$a(x,y) > 0, \; b(x,y) > 0$ and a, b, Φ smooth. Assume periodicity in each spatial direction; $u(x+1,y,t) \equiv u(x,y,t)$ and $u(x,y+1,t) \equiv u(x,y,t)$.

(a) Write a convergent finite difference scheme which is unconditionally stable and which involves only one dimensional inversions.

(b) Justify your answers.

[8] (10 Pts.) Consider the *biharmonic problem* in a two-dimensional domain Ω with sufficiently smooth boundary,

$$\triangle\triangle u = f \text{ in } \Omega,$$
$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma = \partial\Omega,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ.

(a) Show using a Green's formula that, for any $u \in H^2(\Omega)$ satisfying the above boundary conditions, we have

$$\int_\Omega |\triangle u|^2 dxdy = \int_\Omega \left\{ (u_{xx})^2 + (u_{yy})^2 + (u_{xy})^2 + (u_{yx})^2 \right\} dx\, dy.$$

(b) Give a weak variational formulation of the biharmonic problem and show that this has a unique solution u in an appropriate space of functions that you will specify. Assume that $f \in L^2(\Omega)$. Justify your answers.

(c) Describe a finite element approximation of the problem using P_5 elements and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.

(d) Assume convexity and sufficient regularity of the domain Ω. State a standard error estimate for the approximation.