Qualifying Exam, Fall 2013

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Consider a piecewise linear interpolant $\mathrm{L}(x)$ to $\sin(kx)$, $k \in \mathrm{Z}$, $x \in [0,\, 2\pi]$, based upon $\mathrm{N}+1$ equispaced points x_j, where $x_j = \frac{1}{\mathrm{N}}$.

(a) Give a derivation of an estimate of the smallest integer value N, N^*, depending on k, such that

$$\max_{x \in [0,\, 2\pi]} |\sin(kx) - \mathrm{L}(\mathrm{x})| < .01$$

(b) Does the number of points *per wavelength* required to insure such a bound depend upon the value of k?

[2] (5 Pts.) Consider the $(m_1 + m_2) \times (m_1 + m_2)$ block matrix

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

where $A_{1,1}$ is $m_1 \times m_1$, $A_{1,2}$ is $m_1 \times m_2$, $A_{2,1}$ is $m_2 \times m_1$, and $A_{2,2}$ is $m_2 \times m_2$.

(a) Derive an expression for a block lower triangular matrix L and a block upper triangular matrix U in terms of the block components of A, such that $LA = U$

(b) Consider the system of equations

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Derive an expression for the $m_2 \times m_2$ matrix S and the vector $\tilde{f}$ in terms of the block components of A and components of f such that

$$Sx_2 = \tilde{f}$$

is the set of equations that determines the x_2 component of the solution of the original system. *The equations you derive should not include* x_1.

[3] (5 Pts.) Find the approximation of the integral $\int_{-1}^{1} f(x)dx$ using a Gaussian quadrature formula $\sum_{i=1}^{n} c_i f(x_i)$ with $n = 2$. Give the degree of precision of the approximation.

[4] (5 Pts.) Let $f(0)$, $f(h)$ and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$ and $x = 2h$.

(a) Derive the coefficients c_0, c_1 and c_2 so that

$$Df_h(x) = c_0\, f(0) + c_1\, f(h) + c_2\, f(2h)$$

is as accurate an approximation to $f'(0)$ as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in **(a).**

[5] (10 Pts.) Consider the following ODE method for creating approximate solutions of $\dfrac{dy}{dt} = \mathrm{F}(y)$ with timestep k,

$$y^n \;=\; \frac{4}{3}\, y^{n-1} \;-\; \frac{1}{3}\, y^{n-2} \;+\; k\,\frac{2}{3}\,\mathrm{F}(y^n)$$

(a) Derive an expression for the local truncation error.

(b) Show that this method satisfies the root-condition.

(c) Is this a convergent method, and, if it is, what is the global order of accuracy of this method?

(d) Derive the conditions that determine the region of absolute stability for this method.

[6] (10 Pts.) Consider the equation

$$u_t + (u^2 + 1)u_x = \epsilon u_{xx}$$

to be solved for

$$0 \le x \le 1, \quad t \ge 0$$

for $\epsilon > 0$, with $u(x, 0) = u_0(x)$ smooth.

(a) Set up a well posed problem by imposing boundary conditions at $x = 0$ and $x = 1$.

(b) Write a finite difference equation that will remain stable and convergent for all $\epsilon > 0$ and $t > 0$.

(b) Justify your answers.

[7](10 Pts.) Consider the equation

$$u_{tt} = u_{xx} + u_{yy} - 2cu_{xy}$$

to be solved for $0 \le x, y \le 1,\ t > 0$, with smooth initial data

$$\begin{aligned} u(x,y,0) &= \varphi(x,y) \\ u_t(x,y,0) &= \psi(x,y) \end{aligned}$$

with periodic boundary conditions and parameter c.

(a) Write a convergent finite difference scheme for this and choose the appropriate range of values for c.

(b) Justify your answer.

[8] (10 Pts.) Consider the problem in two dimensions,

$$\begin{aligned} -\triangle u + u &= f(x,y), \quad (x,y) \in T, \\ u &= g_1(x), \quad (x,y) \in T_1, \\ u &= g_2(y), \quad (x,y) \in T_2, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x,y)|\ x > 0,\ y > 0,\ x + y < 1\} \\ T_1 &= \{(x,y)|\ y = 0,\ 0 < x < 1\} \\ T_2 &= \{(x,y)|\ x = 0,\ 0 < y < 1\} \\ T_3 &= \{(x,y)|\ x > 0,\ y > 0,\ x + y = 1\}. \end{aligned}$$

(a) Find the weak variational formulation of the problem and verify the assumptions of the Lax-Milgram Lemma by analyzing the appropriate bilinear and linear forms (impose the weakest necessary assumptions on the functions f, g_1, g_2 and h).

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.