Qualifying Exam, Spring 2013

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Consider the linear system $Ax = b$ with $x, b \in \mathbb{R}^n$ and $A = M - N \in \mathbb{R}^{n\times n}$ is nonsingular.

(a) If M is nonsingular and if $(M^{-1}N)^k \to 0$ as $k \to \infty$, show that the iterates x_k, defined by

$$Mx_{k+1} = Nx_k + b,$$

converge to $x = A^{-1}b$ for any starting vector x_0.

(b) Find a splitting $A = M - N$ for the matrix $A = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$, so that the iteration in **(a)** is convergent. Justify your answer.

[2] (5 Pts.) Let $g \in C([a, b])$, with $a \le g(x) \le b$ for all $x \in [a, b]$. Prove the following:

(a) g has at least one fixed point p in the interval $[a, b]$.

(b) If there is a value $0 < \gamma < 1$ such that

$$|g(x) - g(y)| \le \gamma|x - y|$$

for all $x, y \in [a, b]$, then the fixed point p is unique, and the iteration

$$x_{n+1} = g(x_n)$$

converges to p for any initial guess $x_0 \in [a, b]$.

[3] (5 Pts.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function.

(a) For $(x, y) \in [0, \delta x] \times [0, \delta y]$ derive the bilinear interpolation formula for $u(x, y)$ that uses the function values $u(0, 0)$, $u(\delta x, 0)$ and $u(0, \delta y)$, $u(\delta x, \delta y)$ (e.g. the formula that results when you linearly interpolate in one direction followed by linear interpolation in the other direction).

(b) Derive the leading term of error expansion for the error in the interpolated value when using the formula in **(a)**.

[4] (5 Pts.) Let Δ_h be the following three point difference operator that approximates $\dfrac{\mathrm{d}^2u}{\mathrm{dx}^2}$ using a mesh spacing h e.g.

$$\Delta_h u = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

(a) Derive the combination of Δ_h and Δ_{2h} that yields a 4th order approximation to $\dfrac{\mathrm{d}^2u}{\mathrm{dx}^2}$.

(b) Give a derivation of the leading term of the local truncation error for the difference approximation you obtained in **(a)**.

[5] (10 Pts.) Consider the following general 2-stage explicit Runge-Kutta method for advancing the solution of $\dfrac{dy}{dt} = \mathrm{F}(y)$ with timestep k,

$$\begin{aligned} y^* &= y^n \; + \alpha\, k\, \mathrm{F}(y^n) \\ y^{n+1} &= y^n \; + \beta\, k\, \mathrm{F}(y^n) + \gamma\, k\, \mathrm{F}(y^*) \end{aligned}$$

(a) Derive conditions on the coefficients α, β, and γ that insure that the method converges to at least *first* order.

(b) Assuming that the coefficients of the method are selected so that it is first order, derive the expression that determines the interval of absolute stability for the method.

(c) Show that there is at least one set of values $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, so that the resulting method is first order accurate and has an interval of absolute stability that is larger than $[-2, 0]$ (the latter being the interval of absolute stability for all second order methods of the given form).

[6] (10 Pts.) Consider the equation

$$u_t = b_1 u_{xx} + b_2 u_{yy}$$

b_1, b_2 positive constants, to be solved for $0 \leq x, y \leq 1$ with periodic boundary conditions $u(x+1, y, t) \equiv u(x, y, t) \equiv u(x, y+1, t)$ and $u(x, y, 0) = \Phi(x, y)$ given.

(a) Write an unconditionally stable scheme (in time) to solve this equation which involves only one dimensional inversions and which is second order accurate in space.

(b) Justify your answer.

[7](10 Pts.) Consider the equation

$$u_{tt} = au_{xx} + 2bu_{xy} + cu_{yy}$$

to be solved for $t > 0 \;\; 0 \leq x, y \leq 1$ and periodic boundary conditions $u(x+1, y, t) \equiv u(x, y, t) \equiv u(x, y+1, t)$ with initial data $u(x, y, 0), u_t(x, y, 0)$ given.

(a) For what value of the constants a, b, c is the problem well posed?

(b) Write a convergent difference scheme for this problem for choices of the coefficients that result in a well posed problem.

(c) Justify your answers.

[8] (10 Pts.) Consider the problem in two dimensions,

$$\begin{aligned} -\text{div}\Big(\alpha(x)\nabla u\Big) + \beta(x)u &= f(x), \quad & x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \\ u &= 0, & x \in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 5, & x \in \partial\Omega_2, \end{aligned}$$

where $\Omega = \{x \in \mathbb{R}^2 | \; x_1 > 0, \; x_2 > 0, \; x_1 + x_2 < 1\}$,
$\partial\Omega_1 = \{x \in \mathbb{R}^2 | \; x_1 = 0, \; 0 \leq x_2 \leq 1\} \cup \{x \in \mathbb{R}^2 | \; x_2 = 0, \; 0 \leq x_1 \leq 1\}$,
$\partial\Omega_2 = \{x \in \mathbb{R}^2 | \; x_1 > 0, \; x_2 > 0, \; x_1 + x_2 = 1\}$,
$0 < a \leq \alpha(x) \leq A$, $0 < b \leq \beta(x) \leq B$,

with α and β smooth functions and $f \in L^2(\Omega)$.

(a) Find the weak variational formulation and show that the problem is well-posed by verifying the assumptions of the Lax-Milgram Lemma and by analyzing the appropriate bilinear and linear forms.

(b) Develop and describe a piecewise linear Galerkin finite element approximation of the problem that uses a set of basis functions for which the corresponding linear system will be sparse. Show that this linear system has a unique solution.