Qualifying Exam, Fall 2014

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Consider a smooth function $F(\vec{x}) : \mathrm{R}^N \to \mathrm{R}^N$.

(a) Give a derivation of Newton's method for determining solutions of $F(\vec{x}) = \vec{b}$.

(b) If F is a non-singular linear function, derive the number of Newton iterations required to obtain a solution to $F(\vec{x}) = \vec{b}$.

(c) How does your result in (b) depend upon the initial iterate $\vec{x}_0$? Explain.

[2] (5 Pts.) Consider the four pairs of data values (0,0),(2,1),(3,2),(4,3) obtained as samples (x_i, y_i) of a function $y = f(x)$.

(a) Using these values, determine a cubic polynomial that interpolates the inverse function $g(x) = f^{-1}(y)$.

(b) Give the approximate value of $f^{-1}(\frac{1}{2})$ that can be obtained using this interpolant.

[3] (5 Pts.) The forward-difference formula can be expressed as

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

(a) Prove the above formula when f is a smooth function.

(b) Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.

[4] (5 Pts.) Let $t > 0$ and $f(x) = e^{tx}$. We call P_n the Lagrange polynomial, of degree at most $n-1$, which agrees with f at the points 1, 2, ...,n.

(a) Give the absolute error $|f(0) - P_n(0)|$ and prove that

$$t^n \leq |f(0) - P_n(0)| \leq t^n e^{tn}.$$

(b) Discuss, in each of the following cases, whether we gain in using large values for n or not when approximating $f(0)$ by $P_n(0)$:

- $t = 2$
- $t = 0.1$

What do you conclude?

[5] (10 Pts.) Let $f(y)$ be smooth real valued function of y and consider the differential equation

$$\frac{dy}{dt} = f(y), \qquad y(0) = y_0$$

Consider a numerical method that consists of using the Backward Euler method with a timestep Δt and a fixed number, M, of fixed-point iterations to create approximate solutions of the equations that must be solved to advance the solution one timestep. Specifically, to determine y_{n+1} from y_n one uses

(i) $\tilde{y}^0 = y_n$

(ii) $\tilde{y}^k = y_n + \Delta t f(\tilde{y}^{k-1})$ for $k = 1, 2, \ldots M$

(iii) $y_{n+1} = \tilde{y}^M$

(a) Derive interval of absolute stability for the method when $M = 2$.

(b) How does the interval of absolute stability determined in (a) compare with that of Backward Euler if one assumes that the implicit equations are solved exactly?

(c) How does the interval of absolute stability determined in (a) compare with that of Forward Euler?

(d) For general f, derive constraints on the timestep in terms of f and it's derivatives that will insure that the fixed point iteration, (ii), will converge as $M \to \infty$.

(e) Briefly comment on the advisability or inadvisability of using Backward Euler in combination with fixed point iteration to create approximation solutions of stiff ODE's.

[**6**] **(10 Pts.)** Consider the equation

$$u_t = u_{xx} + u_{yy}$$

to be solved in the region $0 \le x \le 1, \quad 0 \le y \le 1$ for $t > 0, \quad u(x,y,0) = \varphi(x,y)$ given and smooth and with periodic boundary conditions:

$$\begin{aligned} u(x+1,y,t) &\equiv u(x,y,t) \\ u(x,y+1,t) &\equiv u(x,y,t) \end{aligned}$$

Assume a finite difference discretization is used with a uniform mesh width $h = \frac{1}{N}$ in each direction and grid points $(x_j, y_k) = (jh,\, kh)$, $j,k = 1 \ldots N$. Denoting $u^n_{j,\,k} = u(x_j,\, y_k,\, t^n)$, let

$$\begin{aligned} D^x_+ D^x_- u_{j,\,k} &= \frac{u^n_{j+1,\,k} - 2u^n_{j,\,k} + u^n_{j-1,\,k}}{h^2} \\ D^y_+ D^y_- u_{j,\,k} &= \frac{u^n_{j,\,k+1} - 2u^n_{j,\,k} + u^n_{j,\,k-1}}{h^2} \end{aligned}$$

Consider the three following splitting methods

[**A**] $\quad u^{n+1} = (I - \frac{\Delta t}{2} D^x_+ D^x_-)^{-1} (I + \frac{\Delta t}{2} D^y_+ D^y_-)(I - \frac{\Delta t}{2} D^y_+ D^y_-)^{-1} (I + \frac{\Delta t}{2} D^x_+ D^x_-) u^n$

[**B**] $\quad u^{n+1} = (I - \Delta t D^x_+ D^x_-)^{-1} [(I - \Delta t D^y_+ D^y_-)^{-1} (I + \Delta t D^x_+ D^x_-) - \Delta t D^x_+ D^x_-] u^n$

[**C**] $\quad u^{n+1} = (I - \Delta t D^y_+ D^y_-)^{-1} (1 + \Delta t D^x_+ D^x_-) u^n.$

[**6a**] For what values of $\lambda = \dfrac{\Delta t}{h^2}$ are each of these stable and convergent?

[**6b**] What is the rate of convergence for each?

Explain your answers.

[**7**] **(10 Pts.)** [**a**] For what values of the constants a, b, c can you obtain stable and convergent finite difference approximations to

$$u_{tt} = a\, u_{xx} + 2b\, u_{xy} + c\, u_{zy}$$

to be solved for $t > 0$ in the region

$$0 \le x, y \le 1$$

with initial conditions

$$\begin{aligned} u(x,y,0) &= \varphi(x,y) \\ u_t(x,y,0) &= \psi(x,y) \end{aligned}$$

φ, ψ smooth, and periodic boundary conditions

$$u(x+1,y,t) \equiv u(x,y,t), u(x,y,+1,t) \equiv u(x,y,t)$$

[**7b**] Give such a scheme.

Justify your answers.

[8] **(10 Pts.)** The following elliptic problem is approximated by the finite element method,

$$
\begin{aligned}
-\text{div}\Big(a(x)\nabla u(x)\Big) &= f(x),\ x \in \Omega, \\
u(x) &= 2,\ x \in \partial\Omega_1, \\
\frac{\partial u(x)}{\partial x_1} + u(x) &= 0,\ x \in \partial\Omega_2, \\
\frac{\partial u(x)}{\partial x_2} &= 0,\ x \in \partial\Omega_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= \{(x_1, x_2):\ 0 < x_1 < 1,\ 0 < x_2 < 1\}, \\
\Gamma_1 = \partial\Omega_1 &= \{(x_1, x_2):\ x_1 = 0,\ 0 \le x_2 \le 1\}, \\
\Gamma_2 = \partial\Omega_2 &= \{(x_1, x_2):\ x_1 = 1,\ 0 \le x_2 \le 1\}, \\
\Gamma_3 = \partial\Omega_3 &= \{(x_1, x_2):\ 0 < x_1 < 1,\ x_2 = 0,\ 1\},
\end{aligned}
$$

and

$$0 < A \le a(x) \le B.$$

(a) Determine an appropriate weak formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness (assume $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$).

(c) Describe a finite element method using P_1 elements, and a set of basis functions such that the linear system thus obtained from the finite element approximation is sparse and of band structure.