Qualifying Exam, Spring 2014

NUMERICAL ANALYSIS

WRITE YOUR STUDENT ID NO. ON EACH PAGE OF YOUR EXAM.
DO NOT WRITE YOUR NAME.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

---

[1] (**5 Pts.**) Consider the following finite difference approximation to $\frac{d^2u}{dx^2}$ on a non-uniform mesh with mesh spacing $h_i = x_i - x_{i-1}$.

$$\frac{d^2u}{dx^2}\Big|_{x_i} \approx \frac{\frac{(u_{i+1}-u_i)}{h_{i+1}} - \frac{(u_i - u_{i-1})}{h_i}}{\frac{(h_i+h_{i+1})}{2}}$$

(a) Derive the leading term of the error expansion associated with this approximation.

(b) If one defines ϵ_{i+1} so that $h_{i+1} = h_i + \epsilon_{i+1}$, what is the largest size of $|\epsilon_{i+1}|$ that can be used and still result in an approximation that has an order of accuracy the same as that of the approximation with equal sized mesh spacing, $h_{i+1} = h_i$?

[2] (**5 Pts.**) Let $I(h)$ be the values of a numerical procedure depending on a discretization parameter h that approximates a value I as $h \to 0$. Let $P_{\tilde{h}}(h)$ be the linear interpolant of $I(h)$ based upon two values $I(\tilde{h})$ and $I(2\tilde{h})$.

(a) Derive the leading term of an error bound for $|P_{\tilde{h}}(h) - I(h)|$ for $h \in [0, 2\tilde{h}]$.

(b) If $I(h)$ has an aysmptotic error expansion of the form $I(h) - I = c_1\,h + c_2\,h^2 + c_3\,h^3 + \cdots$ derive the leading term of the error bound for $|P_{\tilde{h}}(0) - I|$.

[3] (**5 Pts.**) Suppose that $f \in C[a,b]$ and $f(a)\cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f. Describe the Bisection algorithm for generating p_n, and show the approximation error formula

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when } n \ge 1.$$

[4] (**5 Pts.**) Consider the implicit Euler's method (or the backwards Euler's method)

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

for the ODE $y' = f(x,y)$, with $y(0)$ the initial condition. Derive the region of absolute stability for the method. Given an ODE for which $\frac{\partial f}{\partial y} > 0$, does backwards Euler always give the qualitatively correct solution ? Explain.

[5] **(10 Pts.)** Let $f(y)$ and $g(y)$ be smooth real valued functions of y and consider the differential equation

$$\frac{dy}{dt} = f(y) + g(y), \qquad y(0) = y_0$$

(a) Derive the leading term of the local truncation error for the method,

$$\begin{aligned} y^* &= y^n + k\, f(y^n) \\ y^{n+1} &= y^* + k\, g(y^{n+1}) \end{aligned}$$

(b) Assume one can evaluate the derivatives $\frac{df}{dy}$ and $\frac{dg}{dy}$. Determine additional terms, which may incorporate these derivatives, that can be added to the method in **(a)** and result in a higher order method. Justify your results.

[6] **(10 Pts.)** Consider the initial value problem

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= v_x \end{aligned}$$

to be solved for $0 \leq x \leq 1,\ t \geq 0$, with initial and boundary conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \qquad u(1,t) = u(0,t) \\ v(x,0) &= \psi(x), \qquad v(1,t) = v(0,t) \end{aligned}$$

(i) Can you find a stable convergent finite difference scheme for this problem?

(ii) Explain your answer and give an example of such a scheme if one exists.

(b) Consider the related system

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= \frac{1}{1000} u_x + v_x \end{aligned}$$

with the same initial and boundary conditions.

(i) Can you write a stable consistent finite difference scheme for this problem?

(ii) Explain your answer and give an example of such a scheme if one exists.

[7] **(10 Pts.)** Consider the equation

$$u_t = u_{xx} + u_{yy} + u_{zz} - u$$

to be solved for $t > 0$ on the cube $0 \le x, y, z \le 1$, $u = 0$ on the boundary of the cube and $u(x, y, z, 0) = u_0(x, y, z)$, is smooth.

(a) Devise an unconditionally stable, convergent scheme that involves only inverting $n \times n$ matrices if there are n grid points per direction in the discretization.

(b) What is the order of accuracy of your method?

(c) Justify your answers.

[8] **(10 Pts.)** Let V be a Hilbert space with norm $\|\cdot\|_V$. Suppose that $a(\cdot,\cdot)$ is a symmetric bilinear form on $V \times V$ and L a linear form on V such that

(i) $a(\cdot,\cdot)$ is continuous: there is $\gamma > 0$ such that $|a(v,w)| \le \gamma \|v\|_V \|w\|_V \ \forall\ v,\ w \in V$

(ii) $a(\cdot,\cdot)$ is coercive: there is $\alpha > 0$ such that $|a(v,v)| \ge \alpha \|v\|_V^2 \ \forall\ v \in V$

(iii) L is continuous: there is $\Lambda > 0$ such that $|L(v)| \le \Lambda \|v\|_V \ \forall\ v \in V$.

Consider the following abstract problems:

(M) Find $u \in V$ such that $F(u) = \min_{v \in V} F(v)$, where $F(v) = \frac{1}{2} a(v,v) - L(v)$

(V) Find $u \in V$ such that $a(u,v) = L(v) \quad \forall v \in V$.

(a) Show that problems (M) and (V) are equivalent, i.e., $u \in V$ is a solution of (M) if and only if u is a solution of (V).

(b) If u is a solution to these two problems, show the stability estimate

$$\|u\|_V \le \frac{\Lambda}{\alpha}.$$

(c) If u_1 and u_2 are two solutions of (V), show that $u_1 = u_2$.

(d) Let $u \in V$ be solution of (V) and $u_h \in V_h$ (a finite dimensional subspace of V) be such that $a(u_h, v) = L(v)\ \forall v \in V_h$. Show that

$$\|u - u_h\|_V \le \frac{\gamma}{\alpha} \|u - v\|_V \quad \forall v \in V_h.$$