Qualifying Exam, Fall 2015

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

---

[1] **(5 Pts.)** Let A be an $n \times n$ a positive semi-definite symmetric matrix with a non-trivial null space, e.g. the dimension of $\text{Ker}(A) \neq 0$. Consider the problem

$$A\vec{x} = \vec{b} \qquad \text{(M)}$$

(a) State the condition that guarantees a solution to (M) will exist.

(b) Give a derivation of the condition on the matrix A that insures the iterative method

$$\vec{x}_{k+1} = \vec{x}_k + \left(\vec{b} - A\vec{x}_k\right), \quad k = 0, 1, 2, \ldots.$$

will converge to a solution of (M), when a solution to (M) exists.

[2] **(5 Pts.)** Let $f(x), g(x) : \mathcal{R} \to \mathcal{R}$ be smooth functions and consider the problem of finding a solution to

$$f(x) + g(x) = b$$

(a) Assume one is given an approximate solution value x^{n-1}, derive the formula for the next approximation, x^n, that is obtained by using one step of Newton's method with starting iterate x^{n-1} applied to the problem

$$f(x) + g(x^{n-1}) = b$$

(b) Assume that f is a linear function, under what conditions on f and g would you be able to prove that the iteration $x^{n-1} \to x^n$ defined by part **(a)** will converge? Explain your answer.

[3] **(5 Pts.)** Recall that an $n \times n$ matrix A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, ..., n$.

(a) Show that the linear system $Ax = b$ has a unique solution when A is strictly diagonally dominant.

(b) Discuss the convergence of the Jacobi method for solving the linear system $Ax = b$ when A is such a matrix. Justify your answer.

[4] **(5 Pts.)** Let $f(x) = \ln(x+1)$, $x_0 = 0$, $x_1 = 0.6$ and $x_2 = 0.9$.

(a) Construct an interpolating polynomial of degree at most two to approximate f using the three points (you can use $f(0.6) = 0.47$ and $f(0.9) = 0.6$).

(b) Find an error bound for the approximation.

[5] **(10 Pts.)** Assume $f(y) : \mathcal{R} \to \mathcal{R}$ is a smooth function and consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \qquad y(0) = y_0 \qquad\qquad \text{(DE)}$$

(a) Using expansions about t_n, derive the leading term of the local truncation error for the following method used to create approximate solutions to (DE),

$$y_n = y_{n-1} + \frac{3h}{2} f_{n-1} - \frac{h}{2} f_{n-2}$$

(b) Using **(a)** and the fact that the local truncation error estimate for the second order BDF method has the form

$$y_n = \frac{4}{3} y_{n-1} - \frac{1}{3} y_{n-2} + \frac{2h}{3} f_n - \frac{2}{9} h^3 \frac{d^3 y}{dt^3}(t_n) + O(h^4)$$

derive the leading term of the local truncation error for the method

$$y^* = y_{n-1} + \frac{3h}{2} f_{n-1} - \frac{h}{2} f_{n-2}$$

$$y_n = \frac{4}{3} y_{n-1} - \frac{1}{3} y_{n-2} + \frac{2h}{3} f(y^*)$$

(c) Derive the polynomial whose roots determine the region of absolute stability for the method in **(b)**.

[6] **(10 Pts.)** Consider the initial value problem

$$u_t = -(\frac{u^3}{3})_x + \epsilon\, u_{xx} \qquad \epsilon > 0$$

to be solved for $0 \le x \le 1$ with initial data $u(x,0) = \varphi(x)$, smooth and periodic boundary conditions

$$u(x+1, t) \equiv u(x, t).$$

(a) Write a finite difference scheme that converges uniformly in ϵ as $\epsilon \downarrow 0$ for all $t > 0$.

(b) Justify your answers.

[7] **(10 Pts.)** Consider the initial boundary value problem

$$u_{tt} = u_{xx} - u$$

to be solved for

$$0 < x < 1, \quad t > 0$$

$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$

φ, ψ smooth.

(a) For which constant values a, b, c, d do the boundary conditions

$$au_x + bu_t = 0 \quad \text{at } x = 0$$
$$cu_x + du_t = 0 \quad \text{at } x = 1$$

lead to a well posed problem?

(b) Write a convergent finite difference scheme for these well posed problems.

(c) Justify your answers.

[8] **(10 Pts.)** Consider the problem

$$\begin{aligned} -\Delta u + u = f(x,y) & \qquad (x,y) \in \Omega, \\ u = 0 & \qquad (x,y) \in \partial\Omega_1, \\ \frac{\partial u}{\partial \vec{n}} + u = x & \qquad (x,y) \in \partial\Omega_2, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x,y) : \ x^2 + y^2 < 1\}, \\ \partial\Omega_1 &= \{(x,y) : \ x^2 + y^2 = 1, \ x \le 0\}, \\ \partial\Omega_2 &= \{(x,y) : \ x^2 + y^2 = 1, \ x > 0\}, \end{aligned}$$

and $f \in L^2(\Omega)$.

(a) Determine an appropriate weak variational formulation.

(b) Is the obtained bilinear form symmetric ? If yes, give an equivalent minimization formulation.

(c) Verify conditions on the corresponding linear and bilinear forms needed for existence and uniqueness of the solution to the weak variational formulation.

(d) Assume that the boundary $\partial\Omega$ is approximated by a symmetric polygonal curve. Describe a finite element approximation of the problem using P_1 elements and a set of basis functions. Prove the necessary properties of the obtained linear system and discuss its structure. Give a rate of convergence.