Qualifying Exam, Spring 2015

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let A be an $n \times n$ non-singular matrix with real eigenvalues and consider the iteration:

$$\vec{x}_{k+1} = \vec{x}_k + \alpha\left(\vec{b} - A\vec{x}_k\right), \quad k = 0, 1, 2, \dots.$$

(a) Assume that A has both positive and negative eigenvalues. Show that for any given $\alpha \neq 0$, there exists at least one initial vector $\vec{x}_0$ such that this iteration diverges.

(b) Assume that A has only positive eigenvalues. Derive conditions on α under which the iteration will converge for any $\vec{x}_0$. Show how to choose α so that the spectral radius of $I - \alpha A$ will be the smallest.

[2] (5 Pts.) Let x_0, x_1, x_2, x_3 be four equispaced points that are a distance h apart.

(a) Derive the coefficients α_1 and α_2 so that the "open" integration formula

$$\int_{x_0}^{x_3} f(s)\, ds \approx \alpha_1\, f(x_1)\, h \;+\; \alpha_2\, f(x_2)\, h$$

is as high order as possible.

(b) Give an alternate "open" integration formula for approximating the integral over $[x_0,\, x_3]$ that has the same order as (a) but requires fewer function evaluations.

[3] (5 Pts.) Let $f(0)$, $f(-h)$ and $f(-2h)$ be the values of a real valued function at $x = 0$, $x = -h$ and $x = -2h$.

(a) Derive the coefficients c_0, c_1 and c_2 so that

$$Df_h(x) = c_0 f(0) + c_1 f(-h) + c_2 f(-2h)$$

is as accurate an approximation to $f'(0)$ as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in (a).

[4] (5 Pts.) Consider the function $g(x) = 2^{-x}$ on the interval $[\frac{1}{3}, 1]$.

(a) Show that g has a unique fixed point p on this interval.

(b) Estimate the number of iterations necessary to achieve an accuracy of 10^{-4} when applying the fixed point iteration for approximating p. Does the method converge ? Justify your answers.

[5] (10 Pts.) Let A be an $n \times n$ symmetric matrix and consider the initial value problem

$$\text{(IVP)} \qquad \frac{\mathrm{d}\vec{\mathrm{y}}}{\mathrm{dt}} = \mathrm{A}\,\vec{\mathrm{y}} \qquad \vec{\mathrm{y}}(\mathrm{t}_0) = \vec{\mathrm{y}}_0$$

for $t \in [0, \mathrm{T}]$.

(a) Derive the *implicit* Taylor series method to advance an approximate solution of (IVP) from t_n to $t_{n+1} (= t_n + k)$, that is based upon the approximation of $\vec{y}(t_n)$ given by

$$\vec{y}(t_n) \approx \vec{y}(t_{n+1}) - k\,\frac{d\vec{y}}{dt}(t_{n+1}) + \frac{k^2}{2}\,\frac{d^2\vec{y}}{dt^2}(t_{n+1})$$

(b) What is the order of the local truncation error for the method you derived in (a)?

(c) Derive an error bound for the solution to (IVP) obtained with the method in (a). Specifically, derive a bound for $||\vec{e}_{\mathrm{N}}||_2 = ||\vec{y}(t_{\mathrm{N}}) - \vec{y}_{\mathrm{N}}||_2$ in terms of the timestep $k = \frac{\mathrm{T}}{\mathrm{N}}$.

(d) What conditions (if any) on the timestep k did you need to assume in order for your error bound derivation to be valid?

[6] (10 Pts.) Consider the initial boundary value problem

$$u_{tt} = u_{xx} - u$$

for $0 < x < 1,\ 0 < t$ with initial data

$$u(x, 0) = \varphi(x) \qquad\qquad u_t(x, 0) = \psi(x)$$

and φ, ψ smooth.

(a) For which constant values a, b, c, d do the boundary conditions

$$\begin{aligned} au_x + bu_t &= 0 \quad \text{at } x = 0 \\ cu_x + du_t &= 0 \quad \text{at } x = 1 \end{aligned}$$

lead to a well posed problem?

(b) Write a convergent finite difference scheme for these well posed problems.

(c) Justify your answers.

[7] (10 Pts.) Consider the initial value problem

$$u_t = -\frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) + \epsilon u_{xx}$$

$$\epsilon > 0 \tag{1}$$

to be solved for $0 \le x \le 1$ with initial data $u(x,0) = \varphi(x)$, smooth and periodic boundary conditions

$$u(x+1,t) \equiv u(x,t).$$

(a) Write a finite difference scheme that converges uniformly in ϵ as $\epsilon \downarrow 0$ for all $t > 0$.

(b) Justify your answers.

[8] (10 Pts.) Develop and describe the piecewise-linear Galerkin finite element approximation of

$$\begin{aligned}
-\triangle u + u &= f(x,y), \quad (x,y) \in T, \\
u &= g_1(x), \quad (x,y) \in T_1, \\
u &= g_2(y), \quad (x,y) \in T_2, \\
\frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3,
\end{aligned}$$

where

$$\begin{aligned}
T &= \{(x,y)|\ x > 0,\ y > 0,\ x + y < 1\} \\
T_1 &= \{(x,y)|\ y = 0,\ 0 < x < 1\} \\
T_2 &= \{(x,y)|\ x = 0,\ 0 < y < 1\} \\
T_3 &= \{(x,y)|\ x > 0,\ y > 0,\ x + y = 1\}.
\end{aligned}$$

Find a weak variational formulation of the problem and give the necessary assumptions on the functions f, g_1, g_2, and h. Verify the assumptions of the Lax-Milgram Lemma by analyzing the appropriate linear and bilinear forms. Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution. Give a convergence estimate and quote the appropriate theorems for convergence.