Qualifying Exam, Fall 2016

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let A be an $n \times n$ a positive-definite symmetric matrix. A set of vectors $\{\vec{p}_i\}$ for $i = 1 \ldots n$ are A-orthogonal if they are orthogonal with respect to the A-inner product, specifically, $\langle A\,\vec{p}_i,\, \vec{p}_j \rangle = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. Consider the problem of finding a solution to

$$A\,\vec{x} = \vec{b}$$

(a) If one seeks a solution of the form $\vec{x} = \sum_{i=1}^{n} c_i\,\vec{p}_i$ derive expressions for the coefficients c_i.

(b) If one is given a set of n linearly independent vectors $\{\vec{q}_i\}$ give the formulas for a method that can be used to construct a set of A-orthogonal vectors $\{\vec{p}_i\}$ from the set of vectors $\{\vec{q}_i\}$.

[2] (5 Pts.) Consider using Gauss-Seidel to compute the solution to the following system of equations

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Give the vector $\vec{b}$ and the matrix G that arise when the Gauss-Seidel method is expressed in the form $\vec{x}^{k+1} = \vec{b} + \mathrm{G}\,\vec{x}^{k}$.

(b) Does the Gauss-Seidel iteration converge for this system of equations? Justify your answer.

[3] (5 Pts.) A local truncation error estimate for the forward-difference approximation

$$f'(x_0) \approx \frac{1}{h}\Big[f(x_0+h) - f(x_0)\Big]$$

can be expressed as

$$f'(x_0) = \frac{1}{h}\Big[f(x_0+h) - f(x_0)\Big] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

Use Richardson extrapolation to derive an $O(h^3)$ approximation for $f'(x_0)$.

[4] (5 Pts.) Consider a function $f \in C^2[a,b]$.

(a) Derive Newton's method for approximating a zero $p \in [a,b]$ of this function, starting with an initial approximation $p_0 \in [a,b]$. Can you state conditions under which Newton's method will converge at least quadratically ?

(b) Consider $f(x) = e^x - x - 1$. Do we expect any difficulty when applying Newton's method to approximate the zero $p = 0$ of f ? If yes, how could this be avoided ? How can we improve the rate of convergence ? Observe that the first nine iterations obtained using Newton's method starting from $p_0 = 1$ are:

$p_0 = 1$,
$p_1 = 0.58198$,
$p_2 = 0.31906$,
$p_3 = 0.16800$,
$p_4 = 0.08635$,
$p_5 = 0.04380$,
$p_6 = 0.02206$,
$p_7 = 0.01107$,
$p_8 = 0.005545$.

[5] (10 Pts.) Assume $f(y) : \mathcal{R} \to \mathcal{R}$ is a twice continuously differentiable function with a global Lipschitz constant K. Consider using Backward Euler to construct approximate solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \qquad y(0) = y_0 \qquad \text{(DE)}$$

for $t \in [0,T]$ using a uniform timestep of size δt.

(a) Derive the leading term of the local truncation error for Backward Euler applied to (DE).

(b) Derive an error bound for the approximate solution obtained with Backward Euler assuming that the implicit equations are solved exactly.

(c) Derive an error bound for the approximation solution obtained with Backward Euler assuming that at each step the residual associated with the solution of the implicit equations is less than a value ϵ.

(d) How you would choose ϵ as you converge to the solution by letting $\delta t \to 0$? Explain your choice.

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[6] (10 Pts.) Consider the initial value problem

$$u_t = u_{xx} + 2b\, u_{xy} + c^2 u_{xx}$$

to be solved for $t > 0,\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi$ with $u(x, y, t)$ periodic in x and y with period 2π.

(a) For what real values of b and c is the initial value problem with smooth, periodic initial data $u(x, y, 0) = u_0(x, y)$ well posed?

(b) Write a stable, convergent finite difference equation for this problem. Justify your answers

[7] (10 Pts.) Consider the initial value problem

$$u_t + u^4 u_x = \epsilon u_{xx}$$

to be solved for $0 \le x \le 2\pi,\ t > 0$ with $\epsilon > 0$, u periodic in x, period 2π, and smooth, periodic initial data $u(x, 0) = u_0(x)$.

Write a stable convergent difference approximation that remains convergent even as $\epsilon \searrow 0$. Justify your answers.

[8] (10 Pts.) Consider the *biharmonic problem* in a two-dimensional domain Ω with sufficiently smooth boundary,

$$\triangle\triangle u = f \text{ in } \Omega,$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma = \partial\Omega,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ.

(a) Formally show using a Green's formula that, for any $u \in H^2(\Omega)$ satisfying the above boundary conditions, we have

$$\int_\Omega |\triangle u|^2 dxdy = \int_\Omega \left\{ (u_{xx})^2 + (u_{yy})^2 + (u_{xy})^2 + (u_{yx})^2 \right\} dxdy.$$

(b) Derive a weak variational formulation of the biharmonic problem and show that this has a unique solution u in an appropriate space of functions that you will specify. Assume that $f \in L^2(\Omega)$. Justify your answers.

(c) Briefly describe a finite element approximation of the problem using P_5 elements and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.

(d) Assume convexity and sufficient regularity of the domain Ω. State a standard error estimate for the approximation.