Qualifying Exam, Spring 2016

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Determine the number of iterations necessary to find a root of $x^3 + 4x^2 - 10 = 0$ with accuracy 1×10^{-3} using the Bisection method with a starting interval $[1, 2]$. Justify your answer.

[2] (5 Pts.) The second column of the table below gives the errors associated with the approximation of the derivative of e^x at $x = 0$ using a centered difference approximation $\dfrac{f(x+h) - f(x-h)}{2h}$. The third column gives the errors associated with the approximation of the integral $\int_0^1 e^x dx$ obtained using the composite trapezoidal rule with panel width h.

h	Error in Derivative	Error in Integral
10^{-1}	-1.6675e-03	-1.4316e-03
10^{-2}	-1.6667e-05	-1.4318e-05
10^{-3}	-1.6667e-07	-1.4319e-07
10^{-4}	-1.6669e-09	-1.4319e-09
10^{-5}	-1.2102e-11	-1.4296e-11
10^{-6}	2.6756e-11	-1.4033e-13
10^{-7}	5.2636e-10	5.7509e-14
10^{-8}	6.0775e-09	4.0767e-13
10^{-9}	-2.7229e-08	9.4146e-14

(a) For h from 10^{-1} to 10^{-4} why do the errors associated with the derivative and the integral decrease by a factor of .01 with each refinement?

(b) Why do the errors grow when $h < 10^{-5}$ for the approximation of the derivative?

(c) Why don't the errors grow as much for the approximation of the integral when h is very small?

(d) Why do the errors for the integral never get any smaller than $O(10^{-14})$?

[3] (5 Pts.) The Midpoint rule for numerical integration, with error term, is given below,

$$\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi),$$

where $f \in C^2[x_{-1}, x_1]$, $x_{-1} < \xi < x_1$, and $x_0 - x_{-1} = x_1 - x_0 = h > 0$.

Assume that the interval $[0, 2]$ is decomposed into $2n$ sub-intervals of length $h = \frac{1}{n}$. Determine the value of n required to approximate $\int_0^2 e^{2x} \sin 3x \, dx$ with an accuracy of 1×10^{-4} using the Composite Midpoint rule. Justify your answer.

[4] (5 Pts.) Let $u(x)$ and $a(x)$ be smooth functions. Determine the order of accuracy of

$$\frac{(a_{i+1} + a_i)(u_{i+1} - u_i) - (a_i + a_{i-1})(u_i - u_{i-1})}{2h^2}$$

as an approximation to $\left.\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)\right|_{x_i}$ where h is the mesh width, $x_i = ih$, $a_i = a(x_i)$, and $u_i = u(x_i)$.

[5] (10 Pts.) Consider the initial value problem

$$\text{(IVP)} \qquad \frac{dy}{dt} = -\alpha\, y + f(y) \quad y(0) = y_0$$

where $\alpha > 0$ and $|\frac{df}{dy}| < \beta$.

(a) What value of λ associated with the model problem $\frac{dy}{dt} = \lambda y$ should be used when estimating an acceptable timestep for the numerical solution of (IVP)?

(b) Assume that Forward Euler is used to create approximation solutions to (IVP). Give an estimate of the largest timestep that should be used when seeking a qualitatively accurate solution.

(c) If $y(t)$ is a solution of (IVP), let $z(t)$ be defined by the relation $y(t) = \mathrm{e}^{-\alpha t} z(t)$. Derive the initial value problem that $z(t)$ satisfies.

(d) Assume that Forward Euler is used to create approximation solutions to the initial value problem for $z(t)$. Give an estimate of the largest timestep that should be used when seeking a qualitatively accurate solution.

(e) For what values of α and β would it be advisable solve the initial value problem for $z(t)$ rather than the initial value problem for $y(t)$? Explain.

[6] (10 Pts.) Consider the system of partial differential equations

$$u_t + Au_x = 0$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

to be solved for $0 \leq x \leq 1$ and $t \geq 0$ with smooth initial data $u(x, 0) = u_0(x)$ and boundary conditions at $x = 0$ and $x = 1$. The enteries of A, $a_{i,j}$, are constant real values.

(a) What conditions on A and what boundary conditions are needed for well-posedness of this problem?

(b) Give a stable, convergent numerical approximation to this initial value problem.

Justify your statements

[7] (10 Pts.) Consider the initial value problem

$$u_t = -u^2 u_x + \epsilon u_{xx}$$

for $\epsilon > 0$, to be solved for $0 \leq x \leq 1,\ t > 0$ with smooth initial data

$$u(x, 0) = u_0(x)$$

and periodic boundary conditions

$$u(x + 1, t) \equiv u(x, t)$$

(a) Construct a second order accurate convergent method.

(b) Construct a method which remains convergent as $\epsilon \searrow 0$.

Justify your statements

[8] (10 Pts.) The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\nabla \cdot \Big(a(x)\nabla u(x)\Big) &= f(x),\ x \in \Omega, \\ u(x) &= u_0(x),\ x \in \Gamma_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0,\ x \in \Gamma_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0,\ x \in \Gamma_3, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x_1, x_2):\ 0 < x_1 < 1,\ 0 < x_2 < 1\}, \\ \Gamma_1 &= \{(x_1, x_2):\ x_1 = 0,\ 0 \le x_2 \le 1\}, \\ \Gamma_2 &= \{(x_1, x_2):\ x_1 = 1,\ 0 \le x_2 \le 1\}, \\ \Gamma_3 &= \{(x_1, x_2):\ 0 < x_1 < 1,\ x_2 = 0,\ 1\}, \end{aligned}$$

$$0 < A \le a(x) \le B, \quad a.e. \text{ in } \Omega, \quad f \in L^2(\Omega),$$

and $u_0|_{\Gamma_1}$ is the trace of a function $u_0 \in H^1(\Omega)$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Setup a finite element approximation using P_1 elements, and a set of basis functions such that the associated linear system is sparse and of band structure. Discuss the properties of the linear system thus obtained and give the rate of convergence for the approximation.