DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let A be a symmetric tri-diagonal matrix with diagonal elements α_i for $i = 1 \ldots N$ and off diagonal elements β_i for $i = 1 \ldots N - 1$.

(a) Give a derivation of the formulas in terms of the α_i's and β_i's that determine the elements of the LL^t factorization (Choleski factorization) of the matrix A.

(b) What additional conditions on the matrix A will insure that the Choleski factorization exists?

[2] (5 Pts.) Consider using the Trapezoidal method to approximate each of the set of N integrals

$$g(x_n) = \int_0^{x_n} f(s)ds$$

where the $x_n = nh$, $n = 1 \ldots N$ and $h = \frac{1}{N}$.

(a) The cumulative computational work required to evaluate this set of integrals is proportional to what power of N – i.e. what is the order of the computational cost with respect to N?

(b) Give a procedure based upon the Trapezoidal rule that determines approximations to all these integrals with a cumulative computational cost whose order with respect to N is less than that given in **(a)**.

(c) Describe a procedure that gives approximations with an accuracy of $O(h^4)$ and whose computational cost is of the same order with respect to N as the method in **(b)**.

[3] (5 Pts.) Consider the function $f(x) = \frac{1}{x}$ on the interval $[2, 4]$. Find the Lagrange interpolation polynomial that approximates f using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Give the error formula for this polynomial, and, using this error formula, determine a good upper bound for the error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

[4] (5 Pts.) Consider the error formula

$$f'(x_0) - \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] = \frac{h^2}{6}f^{(3)}(\xi)$$

for a centered difference approximation to the derivative of a function f at the point x_0, assuming that f is three times continuously differentiable.

(a) Assume that the evaluation $f(x_0 + h)$ and $f(x_0 - h)$ incurs round-off errors $e(x_0 + h)$ and $e(x_0 - h)$. In the computation of the centered difference the values used are therefore $\tilde{f}(x_0 + h)$ and $\tilde{f}(x_0 - h)$; values related to the true values $f(x_0 + h)$ and $f(x_0 - h)$ by

$$f(x_0 + h) = \tilde{f}(x_0 + h) + e(x_0 + h) \text{ and } f(x_0 - h) = \tilde{f}(x_0 - h) + e(x_0 - h).$$

Assuming that the round-off errors are bounded by some constant $\varepsilon > 0$ and that the third order derivative of f is bounded by a number $M > 0$, show that

$$\left|f'(x_0) - \frac{\tilde{f}(x_0 + h) - \tilde{f}(x_0 - h)}{2h}\right| \leq \frac{\varepsilon}{h} + \frac{h^2}{6}M.$$

(b) Is it reasonable to let h be very small in practice? Why or why not ?

(c) Consider the function $f(x) = \sin x$. Determine an optimal choice for h that minimizes the combined error $e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6}M$ for this function.

[5] (10 Pts.) Consider the following method

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left(y'_{n+1} + y'_n\right) - \frac{\Delta t^2}{12}\left(y''_{n+1} - y''_n\right)$$

to create an approximate solution of the differential equation

$$y' = f(y) \qquad y(0) = y_0 \qquad\qquad \text{(DE)}$$

for $t \in [0, T]$ using a uniform timestep of size $\Delta t = T/N$ where N is the number of timesteps. It is assumed $f(y)$ and its derivatives are known analytically.

(a) Derive the leading term of the local truncation error of this method.

(b) Determine the portion of the negative real axis that is contained within the region of absolute stability of this method.

(c) Assume that $f(y)$ and its derivatives up to order two are bounded for all $y \in \mathcal{R}$. Derive the relation between the error at time t_{n+1}, $e_{n+1} = y(t_{n+1}) - y_{n+1}$, and the error at time t_n, $e_n = y(t_n) - y_n$.

[6] (10 Pts.) Consider the system of differential equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0$$

to be solved for $t > 0,\ 0 \leq x \leq 1$ with smooth initial data

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix}$$

(a) For what values of the constants a, b, c, d will boundary conditions of the form

$$\mathrm{a}\,u(0,t)\ +\ \mathrm{b}\,v(0,t) = 0$$
$$\mathrm{c}\,u(1,t)\ +\ \mathrm{d}\,v(1,t) = 0$$

give a well posed problem?

(b) Give a convergent finite difference scheme that can be used to approximate the solution to this problem. Justify your answers.

[7] (10 Pts.) Consider the initial boundary value problem

$$u_t = \left(x - \frac{1}{2}\right) u_x + \epsilon\, u_{xx}$$

with $\epsilon > 0$ to be solved for $0 \leq x \leq 1,\ t > 0$, u periodic in x of period 1, and smooth periodic initial data $u(x,0) = \phi(x)$.

(a) Give an explicit finite difference scheme to approximate a solution to this problem that is stable and convergent with a timestep restriction of the form

$$\Delta t \leq \mathrm{a}\,\Delta x + \mathrm{b}\,\epsilon\,(\Delta x)^2$$

with $0 < \mathrm{a}, \mathrm{b}$ fixed and independent of ϵ and Δx.

(b) Show that this method is also stable and convergent in the maximum norm.

[8] (10 Pts.) Consider the following problem in a bounded domain $\Omega \subset R^2$:

$$-\mu \triangle u + \alpha_1 \frac{\partial u}{\partial x_1} + \alpha_2 \frac{\partial u}{\partial x_2} + u = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \Gamma = \partial \Omega,$$

where μ and α_1, α_2 are constants with $\mu > 0$.

(a) Derive a weak variational formulation of the problem on an appropriate space of functions. Is there an associated minimization formulation ?

(b) Assuming the appropriate condition on the function f that you will specify, analyze the assumptions of the Lax-Milgram theorem that ensure existence and uniqueness of a weak solution.

(c) Briefly setup a piecewise-linear Galerkin finite element approximation for this problem. Show that the obtained system has a unique solution. Give a convergence estimate and quote the appropriate theorems of convergence.