Qualifying Exam, Spring 2017

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) (a) Let $f : \mathrm{R} \to \mathrm{R}$ be a smooth function with an isolated minimum at a point b and both $f(b)$ and $f''(b)$ are $\mathrm{O}(1)$. Let x_n be an approximation to the point b and assume one has the bound $|f(x_n) - f(b)| < \epsilon$. Derive the leading term for an estimate of $|x_n - b|$ in terms of ϵ.

(b) Suppose that due to numerical errors in evaluating the function f you make the assumption that $|f(x) - f(b)| > \beta$ where $\beta = 1.0 \times 10^{-14}$ for all numerically representable values x. How close can you expect to be able to find an approximation x_n to the value b, the point where f obtains its minimum? Explain.

[2] (5 Pts.) Let $f : \mathrm{R} \to \mathrm{R}$ be a smooth function and $\bar{x}$ a fixed point of f, $f(\bar{x}) = \bar{x}$. Consider deterimining the fixed point $\bar{x}$ by evolving the differential equation

$$\text{(DE)} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = f(x) - x$$

to steady state.

(a) Assume that forward Euler with timestep size k is used to approximate the soluton to the differential equation (DE). Derive a relation between $e_{n+1} = |x^{n+1} - \bar{x}|$ and $e_n = |x^n - \bar{x}|$ where x^n is the approximation to $x(n\,k)$.

(b) Assume $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) < \kappa < 0$ for all x. Determine the restriction on the timestep that must be imposed to insure that x^n will converge to the fixed point $\bar{x}$.

[3] (5 Pts.) Consider the boundary-value problem

$$y'' = q(x)y + r(x), \text{ for } a \le x \le b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta.$$

(a) Using standard centered-difference approximations for $y''(x_i)$, derive the system of equations whose solution can be used to approximate the solution to this problem.

(b) What is the expected order of accuracy of your approximation? Explain.

[4] (5 Pts.) For the midpoint rule, I_m, and the Trapezoidal rule, I_T, asymptotic expansions for the error are given by

$$\int_0^h f(s)ds - I_m = \frac{1}{24}f''(0)h^3 + \frac{1}{48}f'''(0)h^4 + \frac{11}{1920}f^{(4)}(0)h^5 + \frac{13}{11520}f^{(5)}(0)h^6 \ldots$$

$$\int_0^h f(s)ds - I_T = -\frac{1}{12}f''(0)h^3 - \frac{1}{24}f'''(0)h^4 - \frac{1}{80}f^{(4)}(0)h^5 - \frac{1}{360}f^{(5)}(0)h^6 \ldots$$

(a) Derive the coefficients of the approximation

$$\int_0^h f(s)ds = a_0 f(0) + a_1 f(\frac{h}{2}) + a_2 f(h)$$

that results in an approximation of greater accuracy than either the midpoint rule or the Trapezoidal rule.

(b) What is the order of your resulting approximation?

[5] (10 Pts.) Consider the two-step method for the initial value problem $y' = f(t, y)$, $y(0) = y_0$,

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}[4y'_{n+1} - y'_n + 3y'_{n-1}]$$

with $y'_n \equiv f(t_n, y_n)$.

(a) Derive the leading order of the truncation error for this method.

(b) Is this a convergent method, and if so, what is the global order of convergence? Explain, and give a derivation of the conditions that support your conclusion.

(c) To use this method you need two starting values y_0 and y_1. Give a procedure for determining the required starting value y_1 so that the global order of convergence is the same as that determined in **(b)**. Explain.

(d) Derive the conditions that determine the region of absolute stability for this method.

[6] (10 Pts.) Consider the initial value problem

$$\begin{aligned} u_t &= 0 \\ v_t &= u_x \end{aligned}$$

to be solved for $0 \leq x \leq 1,\ t \geq 0$, with initial and boundary conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \qquad u(1,t) = u(0,t) \\ v(x,0) &= \psi(x), \qquad v(1,t) = v(0,t) \end{aligned}$$

(a) (i) Can you write a stable, convergent finite difference scheme for this problem?

(ii) Explain your answer and give an example of such a scheme if one exists.

(b) Consider the related system

$$\begin{aligned} u_t &= \frac{1}{1000} v_x \\ v_t &= u_x \end{aligned} \tag{1}$$

with the same initial and boundary conditions.

(i) Can you write a stable consistent finite difference scheme for this problem?

(ii) Explain your answer and give an example of such a scheme if one exists.

[7] (10 Pts.) Consider the differential equation

$$u_t = a\, u_{xx} + 2\, b\, u_{xy} + c\, u_{yy}$$

with a, b, c constants, to be solved for $t > 0$, $0 \leq x, y \leq 1$ with $u(x, y, 0) = \phi(x, y)$ smooth and boundary conditions $u(0, s, t) = u(1, s, t) = u(s, 0, t) = u(s, 1, t) = 0$ for $s \in [0, 1]$.

(a) For what values of a, b and c would you expect good behavior of the solution?

(b) Write a convergent difference approximation to this problem.

(c) Justify your answers

[8] **(10 Pts.)** Consider the elliptic boundary value problem

$$\begin{aligned} -\triangle u + u &= f(x,y), \quad && (x,y) \in \Omega \\ u &= 1 \quad && (x,y) \in \partial\Omega_1 \\ \frac{\partial u}{\partial \vec{n}} + u &= x \quad && (x,y) \in \partial\Omega_2, \end{aligned}$$

where
$\Omega = \{(x,y)|\ x^2 + y^2 < 1\}$,
$\partial\Omega_1 = \{(x,y)|\ x^2 + y^2 = 1,\ x \leq 0\}$,
$\partial\Omega_2 = \{(x,y)|\ x^2 + y^2 = 1,\ x > 0\}$,
and $\vec{n}$ denotes the exterior unit normal to $\partial\Omega$.

(a) Derive a weak variational formulation of the problem.

(b) Assuming the appropriate condition on the function f, analyze the assumptions of the Lax-Milgram theorem that ensure existence and uniqueness of a weak solution.

(c) Setup a piecewise-linear Galerkin finite element approximation for this problem. Show that the obtained system has a unique solution. Give a convergence estimate and quote the appropriate theorems for convergence.