Qualifying Exam, Fall 2018

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Find constants a, b, c and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has the highest accuracy possible.

[2] (5 Pts.) The forward-difference formula for approximating $f'(x_0)$ can be expressed as

$$f'(x_0) = \frac{1}{h}[f(x_0+h) - f(x_0)] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

Derive an $O(h^3)$ approximation formula for $f'(x_0)$.

[3] (5 Pts.) Let A be an $\mathrm{N} \times \mathrm{N}$ real symmetric matrix with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \leq \lambda_{\mathrm{N}}$ and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\ldots \vec{v}_{\mathrm{N}}$.

(a) Derive the linear interpolant $\mathrm{P}(x)$ so that $\mathrm{P}(\lambda_1) = 1$ and $\mathrm{P}(\lambda_{\mathrm{N}}) = 0$

(b) What are the eigenvectors and corresponding eigenvalues of the matrix P(A)?

(c) When the power method is applied to a matrix $\tilde{\mathrm{A}}$ starting from a vector $\vec{w}^0$, it can be shown that the kth iterate $\vec{w}^k$ satisfies

$$\vec{w}^k = \frac{\tilde{\mathrm{A}}^k \vec{w}^0}{||\tilde{\mathrm{A}}^k \vec{w}^0||_2}$$

Assume that a unit length initial vector $\vec{w}^0$ has components in all the eigenvectors of A and $\vec{w}^k$ is the sequence of vectors obtained using the power method applied to the matrix $\tilde{\mathrm{A}} = \mathrm{P(A)}$. What vector does $\vec{w}^k$ converge to? Justify your answer.

[4] **(5 Pts.)** Let I_h be an approximation to the integral $I = \int_a^b f(x)\,dx$ obtained using the composite Trapezoidal method with mesh size h. Assuming f is a smooth function and h "sufficiently small" then an asymptotic error relation $I_h - I \approx Ch^\alpha$ holds where C is a constant that does not depend upon h.

(a) Assume that one has the exact value of an integral, I, and has computed two approximate values I_{h_1} and I_{h_2} with "sufficiently small" mesh sizes h_1 and $h_2 = \frac{h_1}{2}$. Derive the expression one can use to obtain an estimate of α from this data.

(b) Assume that one doesn't have the exact value of an integral, I, but has computed three approximate values I_{h_1}, I_{h_2}, I_{h_3}, with "sufficiently small" mesh sizes h_1, $h_2 = \frac{h_1}{2}$ and $h_3 = \frac{h_1}{4}$. Derive the expression one can use to obtain an estimate of α from this data.

(c) Give an example of a smooth function for which the procedures in (a) and (b) for estimating α will **not** be reliable for any value of the mesh size.

[5] **(10 Pts.)** Consider the initial value problem

$$\frac{dy}{dt} = f(t) \qquad y(t_0) = y_0 \quad (1)$$

where $f(t)$ is a smooth function of t.

(a) Derive the leading term of the local truncation error when the Trapezoidal method is used to advance an approximate solution to (1) from time t_n to time t_{n+1}.

(b) Assume that over an interval [0, T] one has constructed an approximate solution using the Trapezoidal method with a sequence of time steps $\{h_n\}_{n=0}^{N-1}$ (not necessarily uniform) such that $\sum_{n=0}^{N-1} h_n = \mathrm{T}$. Derive a bound for the error $\tilde{e}_n = y(t_n) - y_n$ for all $n = 1 \ldots N$ where $y(t_n)$ is the exact solution at t_n.

(c) At each step of the numerical method, the size of the local truncation error depends upon the size of the timestep used and derivatives of the solution being computed. Assume one has the capability to estimate the magnitude of the derivatives in the leading term of the local truncation error at any time t. How should the timesteps h_n be chosen so that for a prescribed error tolerance ϵ one has an estimated bound $|\tilde{e}_n| < \epsilon$ for all $n = 1 \ldots N$? Justify your result.

Note: The number of timesteps N taken is not specified in advance.

(d) Suggest a method that could be used to determine the magnitude of the derivatives that occur in the leading term of the local truncation error.

[6] (10 Pts.) Consider the hyperbolic equation

$$u_t + sin(x)\, u_x - y\, u_y = 0$$

to be solved for $t > 0$, (x, y) in the square region $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $u(x, y, 0) = \phi(x, y)$.

(a) Boundary conditions on u are imposed to be zero on which sides of the square? Why?

(b) Set up a finite difference approximation that converges to the correct solution. Justify your answer.

[7] (10 Pts.) Consider the equation

$$u_t = u\, u_{xx} + u\, u_x$$

to be solved for $t > 0, 0 < x < 1$, with $u(0, t) = u(1, t)$, $u_x(0, t) = u_x(1, t)$ and $u(x, 0) = \phi(x)$ where $\phi(x)$ is smooth and periodic with period 1.

(a) What restriction on $\phi(x)$ do you need in order to obtain a convergent finite difference scheme? Why?

(b) Given a ϕ that satisfies the restrictions in (a) give a finite difference scheme that can be used to obtain a convergent approximation. Justify your answers.

[8] (10 Pts.) Develop and describe the piecewise linear Galerkin finite element approximation of

$$\begin{aligned} -\text{div}(a(x)\nabla u) + b(x)u &= f(x), \quad & x &= (x_1, x_2) \in \Omega, \\ u &= 0, & x &\in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x &\in \partial\Omega_2, \end{aligned}$$

where

$\Omega = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}$,
$\partial\Omega_1 = \{x|\ x_1 = 0,\ 0 \le x_2 \le 1\} \cup \{x|\ x_2 = 0,\ 0 \le x_1 \le 1\}$,
$\partial\Omega_2 = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$,
$0 < a \le a(x) \le A$, $0 < b \le b(x) \le B$.

(a) Derive a weak variational formulation of the problem.

(b) Assuming the appropriate conditions on the function f, analyze the assumptions of the Lax-Milgram theorem that ensure existence and uniqueness of a weak solution.

(c) Setup a piecewise-linear Galerkin finite element approximation for this problem. Show that the resulting system of equations has a unique solution. Give a convergence estimate and quote the appropriate theorems for convergence.