**Qualifying Exam, Spring 2018**

# Numerical Analysis

**DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.**

**There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.**

**You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.**

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**[1] (5 Pts.)** Let $D_{h_1,h_2}f$ be the approximation to the derivative $f'(0)$ obtained by evaluating the derivative of the polynomial interpolant through the data $(0, f(0))$, $(h_1, f(h_1))$ and $(h_1+h_2, f(h_1+h_2))$ where both $h_1 > 0$ and $h_2 > 0$.

**(a)** Derive the explicit formula for the approximation $D_{h_1,h_2}f$.

**(b)** With $h_2 > 0$ fixed, to what value does $\lim_{h_1\to 0} D_{h_1,h_2}f$ converge?

**(c)** What is the order of convergence of the approximation to the limit value in (b) with respect to $h_1$?

**[2] (5 Pts.)** Consider the initial value problem

$$(1) \qquad \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}\mathrm{t}^2} = 9\,\mathrm{y} \qquad \mathrm{y}(0) = 2 \quad \frac{\mathrm{d}\mathrm{y}}{\mathrm{d}\mathrm{t}}(0) = 0$$

**(a)** Give the first order system initial value problem that is equivalent to (1).

**(b)** Determine the timestep that is necessary to maintain numerical stability when using Forward Euler to approximate the solution of the equivalent system you derived in (a).

**(c)** It is observed that for any timestep the norm of the numerical solution of the equivalent system obtained with Forward Euler grows as $t \to \infty$. Does this imply that the problem is stiff? Explain.

**[3] (5 Pts.)** Let $P_{0,1,\ldots,n} := P_{x_0,x_1,\ldots,x_n}$ be the interpolating Lagrange polynomial of degree at most $n$ through the points $x_0, x_1, \ldots, x_n$ and values $f(x_0)$, ..., $f(x_n)$, such that $P_{0,1,\ldots,n}(x_i) = f(x_i)$.

**(a)** Let $i, j \in \{0, 1, \ldots, n\}$ be two distinct integers. Express $P_{0,1,\ldots,n}$ in terms of $P_{0,\ldots,i-1,i+1,\ldots,n}$ and $P_{0,\ldots,j-1,j+1,\ldots,n}$.

**(b)** Suppose $x_j = j$ for $j = 0, 1, 2, 3$ and it is known that $P_{0,1}(x) = x + 1$, $P_{1,2}(x) = 3x - 1$, and $P_{1,2,3}(1.5) = 4$. Find $P_{0,1,2,3}(1.5)$.

**[4] (5 Pts.)** Derive the coefficients of a Gaussian quadrature formula of the form $\sum_{i=1}^{n} c_i f(x_i)$ with $n = 2$ to approximate the integral $\int_{-1}^{1} f(x)dx$. What is the maximal degree of the polynomial for which this approximation is exact?

**[5] (10 Pts.)** Consider a predictor-corrector method with Forward Euler predictor and Trapezoidal method corrector;

$$y^* = y_n + h\, f(y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\, f(y_n) \; + \; \frac{h}{2}\, f(y^*)$$

to obtain approximate solutions to

$$(2) \qquad \frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{f(y)} \qquad \mathrm{y(t_0) = y_0}$$

**(a)** Assuming $f(y) : \mathbb{R} \to \mathbb{R}$ is smooth, derive the order of the leading term of the local truncation error for this method.

**(b)** Is the interval of absolute stability for this method greater than the interval of absolute stability for Forward Euler? Justify your answer.

**(c)** Derive the relation between $|e_n| = |y(t_n) - y_n|$ and $|e_{n-1}| = |y(t_{n-1}) - y_{n-1}|$ and the local truncation error. You may assume that $f(y)$ has global Lipschitz constant $L$.

**(d)** Derive an error bound for an approximate solution to (2) obtained with this method.

**[6] (10 Pts.)** Consider the equation

$$u_t = (x - 1/2)\, u_x + (y - 1/2)\, u_y$$

to be solved for $t > 0$, $(x, y) \in [0, 1] \times [0, 1]$ with $u(x, y, 0)$ given.

**(a)** On what part of the space boundary should $u$ be specified?

**(b)** Construct a stable, convergent scheme approximating this problem. Justify your answers.

**[7] (10 Pts.)** Consider the equation

$$u_t + (u^2)u_x = 0$$

to be solved for $t > 0, 0 < x < 1$, with $u(x,t)$ periodic, period 1 in $x$, $u(x,0)$ given.

**(a)** Construct a second order accurate scheme that will converge for $0 < t < T$ for some $T$, if $u(x,0)$ is smooth.

**(b)** Construct a scheme that will converge for all $t > 0$.

Why might these be different? Justify your answers

**[8] (10 Pts.)** Let $\Omega$ be an open, bounded and connected subset of $R^2$, with sufficiently smooth boundary. Consider the problem

$$-\frac{\partial}{\partial x}\Big((1 + 2x^2 + 3y^4)u_x\Big) - u_{yy} = f \text{ in } \Omega,$$

$$(1 + 2x^2 + 3y^4)u_x n_x + u_y n_y + \lambda u = g \text{ on } \Gamma = \partial\Omega,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\vec{n} = (n_x, n_y)$ is the outward unit normal to $\partial\Omega$, and $\lambda \geq 0$ is a constant.

(a) Give weak variational formulations of the problem by considering the cases $\lambda = 0$ and $\lambda > 0$. Show that each of these formulations have one and only one solution (under additional conditions on $u$, $f$ or $g$, if necessary, that you will specify).

(b) In the case $\lambda > 0$, describe a FE approximation using $P_1$ elements, and a set of basis functions such that the corresponding linear system is sparse. In particular show that the corresponding finite dimensional problem has a unique solution.

(c) What would be a standard error estimate for (b) with $P_1$ elements function of the meshsize $h$ ? (assuming convexity and sufficient regularity of $\Omega$ and of its boundary $\Gamma$).