Qualifying Exam, Fall 2019

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] Consider using Gauss-Seidel to compute the solution to the following system of equations

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(a) Give the vector $\vec{b}$ and the matrix G that arises when the Gauss-Seidel method is expressed in the form $\vec{x}^{k+1} = \vec{b} - \mathrm{G}\,\vec{x}^{k}$.

(b) If the initial iterate, $\vec{x}^{\,0} = (0, 0, 0)^t$, what is $\vec{x}^{\,2}$?

(c) Does the Gauss-Seidel iteration converge for this system of equations? Explain why or why not.

[2] (5 Pts.) Give the first order system (in matrix form) that is equivalent to the following second order differential equation. (Be sure to specify the appropriate initial conditions.)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \qquad y(0) = 1 \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 2$$

(b) Give the vector iteration that results when the Trapezoidal method is used to create approximate solutions to the first order system in 2(a).

(c) Describe a solution procedure that can be used to solve the implicit equations in 2(b) to advance the solution and whose success is guaranteed for any choice of stepsize.

[3] (5 Pts.) Consider the function $f(x) = e^x - x - 1$ and the associated fixed point iteration $x^{k+1} = g(x^k)$ where $g(x) = x - \dfrac{2f(x)}{f'(x)}$. What can you say about the rate of convergence of the fixed-point iteration to the root, $\bar{x} = 0$, of f? Justify your statements.

[4] (5 Pts.) Let $f(-h)$, $f(0)$ and $f(h)$ be the values of a differentiable real-valued function f at $x = -h$, $x = 0$, and $x = h$.

(a) Derive the coefficients c_-, c_0 and c_+ so that

$$Df_h(x) = c_- f(-h) + c_0 f(0) + c_+ f(h)$$

is as accurate an approximation to $f'(0)$ as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in **(a)**.

[5] (10 Pts.) Consider an initial value problem of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha\, y + f(y) \qquad y(t_0) = y_0 \qquad (5.1)$$

where $\alpha << -1$ and f is smooth and $-1 \leq \dfrac{\mathrm{d}f}{\mathrm{d}y} \leq 0$ for all $y \in \mathrm{R}$.

(a) Assume one uses forward Euler with a uniform timestep size, h, to advance the solution to (5.1). Derive a timestep constraint based on the specific interval of absolute stability for Euler's method that can be used to estimate the largest timestep size which, if used, will result in an approximate solution in "qualitative agreement" with the exact solution.

(b) Let $y(t)$ be a solution to (5.1) with initial conditions specified at some $t^* \geq t_0$, i.e. $y(t^*) = y^*$. Assuming $y(t) = \mathrm{e}^{\alpha(t-t^*)}\,\phi(t)$, derive a differential equation that $\phi(t)$ must satisfy.

(c) Assume that forward Euler with a uniform timestep size, h, is used to advance the solution to the equation for $\phi(t)$ you derived in 5(b). Will the expected timestep to obtain qualitatively correct solutions for $\phi(t)$ be larger or smaller than the timestep required to obtain qualitatively correct solutions for $y(t)$? Justify your answer.

(d) Now consider a method for advancing y^k to y^{k+1} that consists of applying forward Euler to the equation in 5(b) with $\phi^k = y^k$ for one step and then setting $y^{k+1} = \mathrm{e}^{\alpha h}\,\phi^{k+1}$. This method defines implicitly a numerical method to determine y^{k+1} from y^k. Give an explicit representation of this method for y^k and derive the leading term of the local truncation error.

(e) Consider the task of determining accurate solutions to (5.1). One can either apply a numerical method to (5.1) to determine the approximate values of $y(t)$ directly, or one can apply the numerical method as described in (5)(d) and determine at each step ϕ^{k+1} from y^k and then setting $y^{k+1} = \mathrm{e}^{\alpha h}\phi^{k+1}$. If the numerical method is forward Euler, is there a possible advantage to using one procedure over the other procedure? If one is using an explicit high order multistep method, is there a possible advantage of using one procedure over the other procedure? Explain.

[6] (10 Pts.) Consider the scalar differential equation

$$u_t + f(u)_x = 0$$

to be solved for $t > 0$, u periodic with period 2π in x, and $u(x, 0) = u_0(x)$, a given periodic smooth function. When $f(u) = \frac{u^2}{2}$ consider the two step finite difference scheme

$$\hat{u}_j^{n+1} = u_j^n - \lambda(f(u_{j+1}^n) - f(u_j^n))$$

$$u_j^{n+1} = \frac{1}{2}(u_j^n + \hat{u}_j^{n+1}) - \frac{\lambda}{2}(f(\hat{u}_j^{n+1}) - f(\hat{u}_{j-1}^{n+1}))$$

(a) For what values of $\lambda = \frac{\Delta t}{\Delta x}$ does this scheme converge as Δx, $\Delta t \to 0$?

(b) This convergence is generally valid only for a small interval $0 \leq t \leq T$, for some $T > 0$. Why?

(c) What is the rate of convergence?

(d) Design a scheme that converges for all $t > 0$.

Justify your answers.

[7] (10 Pts.) Consider the equation

$$u_t = u_{xx} + u_{yy} + u_{zz}$$

to be solved for $t > 0$ on the cube $0 \leq x, y, z \leq 1$ with $u = 0$ on the boundary of the cube and with given smooth initial conditions $u(x, y, z, 0) = u_0(x, y, z)$.

(a) Derive an unconditionally stable scheme that involves only inverting $n \times n$ matrices if there are n uniformly spaced grid points in each direction.

(b) What is the order of your scheme? Is there a limit to the order of accuracy that can be obtained with such schemes?

Justify your answers.

[8] **(10 Pts.)** The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\nabla \cdot \Big(a(x) \nabla u(x) \Big) &= f(x), \; x \in \Omega, \\ u(x) &= 0, \; x \in \Gamma_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0, \; x \in \Gamma_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0, \; x \in \Gamma_3, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x_1, x_2) : \; 0 < x_1 < 1, \; 0 < x_2 < 1\}, \\ \Gamma_1 &= \{(x_1, x_2) : \; x_1 = 0, \; 0 \le x_2 \le 1\}, \\ \Gamma_2 &= \{(x_1, x_2) : \; x_1 = 1, \; 0 \le x_2 \le 1\}, \\ \Gamma_3 &= \{(x_1, x_2) : \; 0 < x_1 < 1, \; x_2 = 0, \; 1\}, \end{aligned}$$

$$0 < A \le a(x) \le B, \quad a.e. \text{ in } \Omega, \quad f \in L^2(\Omega).$$

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Setup a finite element approximation using P_1 elements and a set of basis functions such that the associated linear system is sparse and of band structure. Discuss the linear system thus obtained, and give the rate of convergence for the approximation.