NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) (a) Show that if $A = M - N$ is singular and M non-singular, then we can never have $\rho(M^{-1}N) < 1$.

(b) Assume $A = M - N$ is singular and M non-singular. If $\vec{b}$ is any vector, and $\vec{x}^0$ is any non-zero vector, will the iteration $M\vec{x}^{k+1} = N\vec{x}^k + \vec{b}$ necessarily diverge? Explain.

[2] (5 Pts.) Consider the composite Trapezoidal rule with the error term,

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + K_1 h^2 + K_2 h^4 + K_3 H^6 + ...,$$

where each K_i is a constant that depends only on $f^{(2i-1)}(a)$ and $f^{(2i-1)}(b)$. Derive the formulas for an $O(h^4)$ approximation to $\int_a^b f(x)dx$ based upon composite Trapezoidal rule approximations obtained with $n = N$ and $n = 2N$ for $N > 1$.

[3] (5 Pts.) Consider the problem of fitting a function of the form

$$f(x) = a\cos(x) + b\sin(x) + c\cos(2x)$$

to the data

x_i	0	$\pi/2$	π	$3\pi/2$
y_i	1	1	1	0

(a) Give, in matrix form, the linear equations for the coefficients that arise from the conditions $f(x_i) = y_i$.

(b) Give the QR factorization of the system of linear equations you derived in (a).

(c) In terms of the QR factors of the matrix from (a), what are the linear equations that must be solved to determine the coefficients a, b, c so that $\sum_{i=1}^{4} |f(x_i) - y_i|^2$ is a minimum?

(d) Give the coefficients a, b, c that minimize $\sum_{i=1}^{4} |f(x_i) - y_i|^2$.

[4] (5 Pts.) Let A be a given positive constant and $g(x) = 2x - Ax^2$.

(a) Show that if fixed-point iteration converges to a non-zero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

(b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval. Justify your answers.

[5] (10 Pts.) (a) Consider an initial value problem of the form

$$\frac{dy}{dt} = \alpha y + f(y) \qquad y(t_0) = y_0 \qquad (5.1)$$

where $\alpha < 0$ and $-1 \leq \dfrac{df}{dy} \leq 0$.

Assume one uses forward Euler with a uniform timestep size, h, to advance the solution to (5.1). Derive a timestep constraint, based on the specific interval of absolute stability for Euler's method, that can be used to estimate the largest timestep size which, if used, will still result in an approximate solution in "qualitative agreement" with the exact solution.

(b) Consider using a standard fixed point iteration to determine the solution to the implicit equations that must be solved at each step when using the Trapezoidal method to advance an approximate solution of (5.1) from (t_k, y_k) to $(t_k + h, y_{k+1})$;

$$z^{n+1} = y_k + \frac{h}{2}g(y_k) + \frac{h}{2}g(z^n)$$

where $g(y) = \alpha y + f(y)$. Derive the restriction on the timestep, h, that is necessary to insure that the iteration z^n for $n = 0, 1, 2, \ldots$ with $z^0 = y_k$ will converge as $n \to \infty$.

(c) Now consider the initial value problem for a linear, constant coefficient system of differential equations

$$\frac{d\vec{y}}{dt} = A\vec{y} \qquad \vec{y}(t_0) = \vec{y}_0 \qquad (5.2)$$

where the matrix A is symmetric, negative definite. If one uses, as in (b), standard fixed point iteration to determine the solution to the implicit equations that must be solved at each step when using the Trapezoidal method to advance an approximate solution of (5.2), derive the timestep restrictions, if any, on the timestep, h, that is necessary to insure that the vector iteration $\vec{z}^n$ for $n = 0, 1, 2, \ldots$ with $\vec{z}^0 = \vec{y}_k$ will converge as $n \to \infty$.

(d) If the problem in (5.2) is stiff and the matrix A is large and sparse, it is common to use the Trapezoidal method in combination with an iterative method to solve the implicit equations. In such a case, should one use standard fixed point iteration to solve the implicit equations, and if not, suggest an alternate iterative method that could be used. Explain your answer.

[6] (10 Pts.) Consider the equation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = 0$$

to be solved for $x \in [0, 1]$, $t > 0$, with $u_1(x, 0) = \phi(x)$ and $u_2(x, 0) = \psi(x)$ given.

(a) What are the most general boundary conditions of the form

$$a\, u_1(0, t) \;+\; b\, u_2(0, t) = 0$$
$$c\, u_1(1, t) \;+\; d\, u_2(1, t) = 0$$

for real constants, a, b, c, and d, that give a well posed problem?

(b) Devise a convergent numerical scheme for the well-posed problems. Justify your answers.

[7] (10 Pts.) Let a, b, c, d, and e be real constants and consider the convection-diffusion equation

$$u_t + a\, u_x + b\, u_y = c\, u_{xx} + d\, u_{xy} + e\, u_{yy}$$

to be solved for $u(x, y, t)$ for $(x, y) \in [0, 1] \times [0, 1]$, $t > 0$, with $u(x, y, 0)$ specified and $u(x, y, t)$ periodic in both the x and y coordinates with period 1.

(a) What conditions on the constants are required to make this problem well posed?

(b) Devise a convergent numerical scheme for the well-posed problems. Justify your answers.

[8] (10 Pts.) Consider the differential equation

$$-\triangle u = f(x, y), \qquad (x, y) \in \Omega$$
$$u = 0 \quad (x, y) \in \partial\Omega_1$$
$$\frac{\partial u}{\partial n} + u = x^2 \quad (x, y) \in \partial\Omega_2,$$

where
$\Omega = \{(x, y) | \ x^2 + y^2 < 1\}$,
$\partial\Omega_1 = \{(x, y) | \ x^2 + y^2 = 1, \ x \le 0\}$,
$\partial\Omega_2 = \{(x, y) | \ x^2 + y^2 = 1, \ x > 0\}$.

(a) Derive a weak variational formulation of the problem.

(b) Give conditions on f, and, assuming these conditions hold, verify that the assumptions of the Lax-Milgram theorem are satisfied, thus ensuring existence and uniqueness of a weak solution.

(c) Describe a piecewise-linear Galerkin finite element approximation for the problem using P_1 elements.

(d) What would be a standard error estimate for **(c)** as function of the meshsize h ?