Qualifying Exam, Spring 2020

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let $f(0)$, $f(h)$ and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$ and $x = 2h$.

(a) Derive the coefficients c_0, c_1 and c_2 so that

$$Df_h(x) = c_0\, f(0) + c_1\, f(h) + c_2\, f(2h)$$

is as accurate an approximation to $f'(0)$ as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in **(a).**

[2] (5 Pts.) Let $f(x) = \sqrt{\pi x} - \cos(\pi x)$.

(a) Show that the equation $f(x) = 0$ has at least one solution p in the interval $[0, 1]$.

(b) When using the Bisection method to approximate p, how many iterations are necessary to solve $\sqrt{\pi x} - \cos(\pi x) = 0$ with accuracy 10^{-5} on $[0, 1]$?

[3] (5 Pts.) If $f(x)$ is sufficiently differentiable, then the error in approximations to $\mathrm{I} = \int_a^b f(x)\, dx$ obtained using the composite trapezoidal method, $\mathrm{I}_T(h)$, and the composite midpoint method, $\mathrm{I}_M(h)$, have asymptotic expansions of the form

$$\mathrm{I} - \mathrm{I}_T(h) = -\frac{h^2}{12}\left(f^{'}(b) - f^{'}(a)\right) + O(h^4)$$

$$\mathrm{I} - \mathrm{I}_M(h) = -\frac{h^2}{24}\left(f^{'}(b) - f^{'}(a)\right) + O(h^4)$$

where h is the mesh width.

(a) For a given value of h, determine the combination of the values of $\mathrm{I}_T(h)$ and $\mathrm{I}_M(h)$ that results in an integral approximation with a higher order rate of convergence.

(b) What are the weights of the integration formula resulting from the combination you derived in **(a)**?

[4] (5 Pts.) Let $f(x)$ be a real valued function and assume $x_1 \neq 0$. Prove that there exists a unique quartic polynomial $q(x)$ so that

$$q(0) = f(0), \qquad q'(0) = f'(0), \qquad q''(0) = f''(0), \qquad q'''(0) = f'''(0),$$
$$q(x_1) = f(x_1).$$

[5] (10 Pts.) Consider the linear system of ODE's

$$\frac{d\vec{y}}{dt} = \mathrm{A}\,\tilde{\mathrm{y}} \qquad \tilde{\mathrm{y}}(\mathrm{t}_0) = \tilde{\mathrm{y}}_0.$$

where A is the $\mathrm{N} \times \mathrm{N}$ symmetric tri-diagonal matrix

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & * & * & * & & \\ & & & * & * & * & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}$$

with $h = \dfrac{1}{\mathrm{N}+1}$.

(a) The eigenvalues of A are given by $\lambda_k = -\dfrac{4}{h^2}\sin^2(\dfrac{k\pi}{\mathrm{N}+1})$ $k = 1\ldots\mathrm{N}$. Give a good estimate for the smallest value of the Lipschitz constant for the function $\vec{\mathrm{F}}(\vec{y}) = \mathrm{A}\vec{y}$ when using the ℓ^2 norm on R^{N}.

(b) Assume approximate solutions of this system of equations are obtained for $t \in [0,1]$ using Euler's method with a uniform timestep of size $\Delta t = \dfrac{1}{M}$. Give a derivation of an error bound for Euler's method, and in particular, derive expressions for the constants, C_1 and C_2 appearing in an error bound of the form

$$|\tilde{e}_n| \leq C_1\,|\tilde{e}_0| + C_2\,\Delta t \qquad n = 1, 2, \ldots N$$

where $\tilde{e}_n = y^n - y(t_n)$. Show your work.

(c) Assuming $\tilde{e}_0 = 0$ and $h = 0.1$, give an estimate of Δt so that the magnitude of the error bound at $t = 1.0$ is less than 1.0×10^{-3}. (One can use the approximation that $\mathrm{e} \approx 10^{.43}$).

(d) Is it necessary to use a timestep of the size determined in **(c)** to obtain an accurate solution? Explain your answer.

[6] (10 Pts.) Consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = c\frac{\partial^2 u}{\partial x^2}$$

to be solved for $0 \le x \le 1$, $t > 0$, with periodic boundary conditions in x and initial data

$$u(x,\, 0) = f(x), \qquad f(x) \textit{ smooth}.$$

Here c is a positive constant.

(a) Construct a stable, convergent finite difference scheme for $c > 0$ that remains convergent even as $c \downarrow 0$. Hint: Differentiate the equation with respect to x and solve for $\dfrac{\partial u}{\partial x} = v$, and use everything you know about the resulting equation for v.

(b) Justify your answer.

[7](10 Pts.) Consider the scalar second order equation for $u(x,t)$

$$a\, u_{tt} \;+\; 2\, b\, u_{xt} \;+\; c\, u_{xx} = 0$$

to be solved for $t > 0$, $0 \le x \le 1$ with periodic boundary conditions in x and initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

with a, b and c constants and $f(x)$ and $g(x)$ smooth.

(a) For what values of a, b and c is this problem well posed?

(b) Devise a convergent finite difference scheme to create approximate solutions to this problem.

(c) Justify your answers.

Hint: For parts (b) and (c), one option might be to make this into an equivalent first order system of equations.

[8] (10 Pts.) Consider the problem in two dimensions,

$$\begin{aligned} -\triangle u + u &= f(x,y), \quad (x,y) \in T, \\ u &= 0, \quad (x,y) \in T_1 \cup T_2, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x,y)|\ x > 0,\ y > 0,\ x+y < 1\} \\ T_1 &= \{(x,y)|\ y = 0,\ 0 < x < 1\} \\ T_2 &= \{(x,y)|\ x = 0,\ 0 < y < 1\} \\ T_3 &= \{(x,y)|\ x > 0,\ y > 0,\ x+y = 1\}. \end{aligned}$$

(a) Find the weak variational formulation of the problem and verify the assumptions of the Lax-Milgram Lemma by analyzing the appropriate bilinear and linear forms (impose the weakest necessary assumptions on the functions f and h).

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution. State the rate of convergence for the approximation.