Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers. Read the directions of the exam very carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1. ______________________ 5. __________________________

2. ______________________ 6. __________________________

3. ______________________ 7. __________________________

4. ______________________ 8. __________________________

Pass/fail recommend on this form.

Total score: ___________________

Form revised 3/08

ADE Exam, Fall 2021
Department of Mathematics, UCLA

1. [10 points] Find all equilibrium points of the following system of differential equations:

$$\frac{dx}{dt} = \sin(x+y)\,,$$
$$\frac{dy}{dt} = e^x - 1\,.$$

Determine whether each equilibrium point is stable or unstable. Also state precisely what the difference is between stability and asymptotic stability.

2. [10 points] Use Frobenius series near $t = 0$ to find two independent solutions to the differential equation

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \left(t^2 - \frac{1}{4}\right)y = 0\,, \quad 0 < t < \infty\,.$$

Subsequently give explicit expressions for the coefficients of your series solutions.

3. [10 points] The concentration $c(r,t)$ of diffusing molecules within a droplet obeys the following PDE:

$$\frac{\partial c}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)\ , \quad r < 2\ , \quad t > 0 \tag{1}$$

along with side conditions

$$c(r,0) = \begin{cases} 0 & \text{if } r < 1 \\ 1 & \text{if } r > 1 \end{cases} \qquad \text{and} \qquad -\left.\frac{\partial c}{\partial r}\right|_{r=2} = 0\ .$$

Here r is the usual spherical polar coordinate, representing distance to the center of the droplet.

By seeking solutions of the form $c(r,t) = \frac{\rho(r)}{r}T(t)$, or otherwise, prove the following facts:

(a) In the limit $t \to \infty$, it is the case that $c(r,t) \to c_\infty$. You should find the constant c_∞.

(b) Also, in the limit $t \to \infty$, it is the case that $|c(r,t) - c_\infty| < Ce^{-\alpha t}$ for some positive constants C and α. You should identify the largest possible value for the constant α.

Note: You do not need to give the value of α explicitly; it is sufficient to define it implicitly as a root of an equation. There is no need to find a value for C.

4. [10 points] Show, for a constant $\beta \geq 0$ is a constant, that the PDE

$$u_{tt} + \beta u_t - u_{xx} + u = 0\ , \quad x \in \mathbb{R} \tag{2}$$

has at most one compactly supported solution, if given $\mathcal{C}^2$ initial data $u(x,0) = \phi(x)$ and $u_{,t}(x,0) = \psi(x)$.

5. [10 points] A function $u(x,t)$ satisfies the nonlinear PDE

$$\Delta u - u^3 = 0 \tag{3}$$

on a bounded, open domain $\Omega \subset \mathbb{R}^d$, with boundary conditions $u = g(x)$ on $\partial\Omega$. Assume that $u \in \mathcal{C}^2(\Omega) \cup \mathcal{C}(\bar{\Omega})$, and that $g(x) > 0$ at some $x \in \partial\Omega$.

(a) Show that $u(x) < \max_{X \in \partial\Omega}\{g(X)\}$ for all $x \in \Omega$.

(b) Consider the special case $d = 1$ and $\Omega = [-1, 1]$. By choosing appropriate boundary conditions at $g(\pm 1)$, show that $u(x)$ can attain values less than $\min\{g(\pm 1)\}$.

6. [10 points] Consider Burgers equation $u_t + uu_x = 0$ in $\{x, t > 0\}$ with initial condition $u(x, 0) = 0$ and boundary condition $u(0, t) = 1$ for $0 < t < 1$ and $u(0, t) = 2$ for $t > 1$.

(a) Plot the characterstics coming from the t-axis.

(b) What is the entropy solution for $0 < t < 1$?
[HINT: it should be related to the Riemann problem]

(c) For $t > 2$ there is a new structure emerging from the t axis. What is it? When will it merge with the structure from part (b)?

(d) Write the full solution to the problem for $t > 0$, satisfying the entropy condition.

7. [10 points] The damped Kuramoto–Sivashinsky equation is

$$u_t + uu_x = -\gamma u_{xx} - u_{xxxx} - \beta u\,,$$

where $\gamma, \beta > 0$ are constants.

Show that two smooth solutions on a periodic interval $[0, L]$ that have the same initial data remain equal to each other at later times. Note that since $\gamma > 0$ the first term on the right hand side is destabilizing. You can break the problem down in two steps:

(a) Given any $\epsilon > 0$ prove there exists a constant C so that any smooth function u on the periodic interval satisfies

$$\left| \int_0^L u u_{xx} dx \right| < \epsilon \int_0^L u_{xx}^2 dx + C \int_0^L u^2 dx.$$

(b) Using the above and Grönwall's Lemma, provide the uniqueness result.

8. [10 points] The Ginzburg–Landau energy is $G(u) = \frac{1}{2}\int |\nabla u|^2 + \int W(u)$, where $W(u) = (1 - u^2)^2$ is a double-well potential with local minima at $u = \pm 1$. The L^2 gradient descent of the energy G is the Allen–Cahn (AC) equation

$$u_t = -\frac{\delta G}{\delta u} = \Delta u - W'(u)\,.$$

Here the first variation uses the L^2 inner product.

(a) Consider the AC equation on the line in 1D. Write the associated steady-state equation as a coupled system of first-order equations for $u(x)$ and $v(x) = u'(x)$. Show that this is a Hamiltonian system by finding the formula for the Hamiltonian, up to a constant.

(b) Sketch the phase portrait of the steady-state solutions in the u-v plane; label all equilibria and separatrices.

Note: The main point here is to get the topology correct rather than proving theorems about the equilibria.

(c) Show that there are periodic solutions for all periods T for $T \geq T_0$ for some T_0 and determine what is T_0.

(d) Show that there are solutions with $u \to \pm 1$ as $x \to -\infty$ and $u \to \mp 1$ as $x \to +\infty$ by pointing out what those correspond to on your phase potrait and why.

(e) Consider the diffuse-interface rescaling of the AC equation:

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} W'(u) \,.$$

Show that there exists a steady-state solution of this equation that asymptotes to 1 as $x \to -\infty$ and to -1 as $x \to +\infty$ and such that the transition between -1 and 1 occurs over a lengthscale of $O(\epsilon)$.