Analysis qualifying exam, Fall 2021

Instructions and rubric

- There are 12 problems: 6 on real analysis, 6 on complex analysis.
- Attempt at most five questions on real analysis and five questions on complex analysis. If you submit answers to more questions than this, please indicate clearly which questions should be graded.
- All questions will be graded out of 10 points. Questions with several parts show the breakdown of points in square brackets.
- In case of partial progress on a problem, details will usually earn more points if they are explained as part of a solution outline for the whole problem.

Conventions

- All Banach spaces are over the reals unless indicated otherwise.
- All functions are real-valued unless indicated otherwise.
- L^p always refers to Lebesgue measure unless a different measure is indicated.
- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of positive integers and real and complex numbers, respectively, and $\mathbb{D}$ is the open unit disk in $\mathbb{C}$.

Real analysis

1. Let $f : [0, 2\pi] \to \mathbb{C}$ belong to L^1 and assume that

$$\int_0^{2\pi} f(x)\Big(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^4\varphi}{\partial x^4}\Big)\, dx = 0$$

whenever $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth and (2π)-periodic. Prove that

$$f(x) = a + be^{ix} + ce^{-ix} \quad \text{a.e.}$$

for some complex scalars a, b, c.

2. Let $f_1, f_2, \dots \in L^1([0, 1])$ satisfy

$$\int_0^1 |f_i|^2\, dx = \infty \quad \text{for every } i.$$

(a) [5 points] Prove that the set

$$A_{i,M} = \Big\{ g \in L^1([0, 1]) : \; M < \int_0^1 |f_i g|\, dx \le \infty \Big\}$$

is open in the norm topology of L^1 for every integer i and every $M > 0$.

(b) [5 points] Prove that some $g \in L^1$ satisfies

$$\int_0^1 |f_i g|\, dx = \infty \qquad \text{for every } i.$$

3. Let $\varphi : [0, 1] \to [0, 1]$ be Borel measurable. Prove that there is a Borel set $B \subseteq \varphi([0, 1])$ such that $m(\varphi^{-1}(B)) = 1$. Here m denotes Lebesgue measure on $[0, 1]$.

Remarks: (i) The image $\varphi([0, 1])$ need not be Borel. (ii) It is possible to choose B to be a countable union of compact sets.

4. Let $r_1 > r_2 > \cdots > 0$. For each positive integer n, let $\mathcal{C}_n$ be a pairwise disjoint collection of 2^n closed disks of radius r_n in $[0, 1]^2$, and assume that every member of $\mathcal{C}_n$ contains exactly two members of $\mathcal{C}_{n+1}$. Let K_n be the union $\bigcup_{D \in \mathcal{C}_n} D$, and let $K = \bigcap_{n=1}^{\infty} K_n$.

(a) [7 points] Prove that there is a Borel probability measure μ such that $\mu(K) = 1$ and $\mu(D) = 2^{-n}$ for every $D \in \mathcal{C}_n$.

(b) [3 points] Prove that K is the support of μ: that is, the smallest closed set whose measure equals 1.

5. Let $1 \le p \le \infty$ and let φ and ψ be nonzero bounded linear functionals on $L^p(\mathbb{R})$. Assume that $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$. For precisely which values of p does this imply that φ and ψ are linearly dependent? Justify your answer.

6. Let K be a continuous function on $\mathbb{R}^2$ that is periodic in both coordinates:

$$K(x+1, y) = K(x, y+1) = K(x, y).$$

Given any $F \in L^1([0,1] \times [0,1])$, show that

$$\int_{[0,1]^2} K(x, y+nx) F(x,y)\, dm(x,y) \to \int_0^1 \Big(\int_0^1 K(x,s)\, ds \Big) \Big(\int_0^1 F(x,y)\, dy \Big)\, dx$$

as $n \to \infty$, where m is two-dimensional Lebesgue measure.

Complex analysis

7. Let f and g be functions that are continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. Suppose that $\mathrm{Re}(f)$ and $\mathrm{Re}(g)$ agree on $\partial\mathbb{D}$. Prove that $f - g$ is an imaginary constant on $\mathbb{D}$.

8. Throughout this question, U, V and W are proper nonempty subsets of $\mathbb{C}$ that are open and simply connected, and u and v are fixed points in U and V respectively. We say that a sequence of functions converges *normally* if it converges uniformly on compact sets.

   (a) [4 points] Prove that, for any compact set $K \subseteq U$, there is a compact set $L \subseteq V$ such that $f(K) \subseteq L$ for any holomorphic map $f : U \to V$ that satisfies $f(u) = v$.

   (b) [6 points] Let $f_1, f_2, \ldots$ be a sequence of holomorphic maps $U \to V$ that all satisfy $f_n(u) = v$ and that converge normally to another holomorphic map $f : U \to V$. Let $g : W \to U$ and $h : V \to W$ be conformal equivalences. Prove that $f_n \circ g$ converges normally to $f \circ g$ and $h \circ f_n$ converges normally to $h \circ f$.

9. Compute the number of solutions, including multiplicity, of the equation $z^5 \cos z + 5iz^4 + 2 = 0$ inside the unit disc $|z| < 1$.

10. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy $|f'(z)| \le 2|f(z)|$ for all $z \in \mathbb{C}$.

11. For each $p \in (-1, 1)$, compute the improper Riemann integral

$$\int_0^\infty \frac{x^p}{x^2+1} dx.$$

12. Let $f(z)$ be a holomorphic function on the set $\mathcal{B} = \{z : |z| < 2\}$ that satisfies $|f(z)| < 1$ for all $z \in \mathcal{B}$. Assume also that

$$f(1) = f(-1) = f(i) = f(-i) = 0.$$

(a) [5 points] Show that $|f(0)| \leq 1/16$.

(b) [5 points] Show that there is a such function $f : \mathcal{B} \to \mathbb{D}$ with $|f(0)| = 1/16$.