Basic Exam: Fall 2021

September 8, 2021

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

The final score will be the sum of **FOUR** analysis problems (Problems 1–5) and **FOUR** linear algebra problems (Problems 6–10). *On the front of your paper indicate which 8 problems you wish to have graded.* Please be reminded that to pass the exam you need to show mastery of both subjects. Little or no credit will be given for answers withour adequate justification. Good Luck.

1	2	3	4	5
6	7	8	9	10

1. Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.
   (a) Show that S converges conditionally.
   (b) Show that S converges to $\ln 2$.
   (c) Show how to rearrange the series S such that it converges to the limit $\sqrt{2}$.

2. Let $f(t)$ be a real valued function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Assume that $f(0)=0$ and $|f'(t)| \leq |f(t)|$ for all $t \in (0,1)$. Prove that for all $t \in (0,1)$ it holds $f(t)=0$.

3. Let a_n be a sequence of real numbers. Proof that the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \int_0^x e^{t^8 - 6\cos^2(a_n t)} dt$$

has a subsequence that converges uniformly on $[0,1]$.

4. Prove that a metric space is sequentially compact if and only if it is complete and totally bounded. A metric space is totally bounded when for every $\varepsilon > 0$ it can be covered by a finite number of balls of radius ε.

5. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ that decreases monotonically to 0, and let $\{b_n\}$ be a sequence with bounded partial sums, i.e. there exists $M \in \mathbb{R}$ such that for all n,

$$\left|\sum_{k=1}^{n} b_k\right| \leq M$$

Prove that $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

6. Let A be a complex $n \times n$ matrix such that $A^2 = A$.
   (a) Prove that A is similar to a diagonal matrix.
   (b) Prove that $\mathrm{tr}(A)$ is a non-negative integer.

7. Let A, B be two $n \times n$ complex matrices such that $AB = BA$. Prove that these matrices have a common eigenvector.

8. Let A, B be complex $m \times n$ matrices such that $A^T B = 0$. prove that $\text{rk}(A) + \text{rk}(B) \leq m$.

9. Let V be a vector space over a field F. If X and Y are subspaces of V such that $V = X \oplus Y$, recall that the *projection onto* X *along* Y is the linear operator $P : V \to V$ defined as follows: for $v \in V$, let $v = x + y$ with $x \in X$, $y \in Y$; then

$$P(v) = x$$

Prove that a linear operator $P : V \to V$ is a projection if and only if $P^2 = P$.

10. Let V be a finite dimensional inner product space over $\mathbb{C}$, and let T be a linear operator on V.

    1. Prove that if T is invertible, then T has a square root, i.e. there exists a linear operator R on V such that $R^2 = T$.
    2. Now assume that T is diagonalizable, but not necessarily invertible. Under what conditions are there only a finite number of distinct square roots of T, and how many are there?