Geometry/Topology Qualifying Exam

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers.

Read the directions of the exam carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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Pass/fail recommend on this form.

Total score: ____________________

Form revised 3/08

QUALIFYING EXAM: GEOMETRY/TOPOLOGY FALL 2021

Instructions: Do all 10 problems. Each problem is worth 10 points.

(Q-1) Let $V_k(\mathbb{R}^n)$ denote the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. Show that $V_k(\mathbb{R}^n)$ is a manifold of dimension $k\left(n-\frac{k+1}{2}\right)$. Hint: Use a map $F\colon M_{n\times k}(\mathbb{R}) \to \mathbb{R}^{k(k+1)/2}$ such that $V_k(\mathbb{R}^n)$ becomes the preimage of a regular value of F. (Here $M_{n\times k}(\mathbb{R})$ denotes the set of matrices with n rows and k columns.)

(Q-2) Show that the product of two spheres $S^p \times S^q$ is parallelizable provided p or q is odd. (Here parallelizable means the tangent bundle is trivializable; equivalently, there exist $(p+q)$ vector fields on $S^p \times S^q$ which are everywhere linearly independent.)

(Q-3) Let $M^m \subset \mathbb{R}^n \setminus \{0\}$ be a compact smooth submanifold of dimension m. Show that M is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. (Here "almost all" means that the set of subspaces that are not transverse to M has measure zero.)

(Q-4) Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactly supported n-form. Show that $\omega = d\eta$ for some compactly supported $(n-1)$-form $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.

(Q-5) Let $n \geq 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

(Q-6) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nowhere zero continuous function. Prove that there exists a continuous function $g\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.

(Q-7) In this problem, work in either the category of topological manifolds or smooth manifolds (your choice). Let M be an n-manifold. Define its orientation double cover $\widetilde{M}$, and explain its structure as a topological/smooth manifold. Prove that the orientation double cover of $\widetilde{M}$ is always disconnected.

(Q-8) Let M be a connected non-orientable manifold whose fundamental group G is simple (that is, has no non-trivial normal subgroup). Prove that G must be isomorphic to $\mathbb{Z}/2$.

(Q-9) Let X be the quotient of the space $\{0,1,2\} \times S^1 \times D^2$ by the relation

$$(0, z_1, z_2) \sim (1, z_1, z_2) \sim (2, z_1, z_2) \qquad \forall z_1, z_2 \in S^1.$$

(Here S^1 is the unit circle and D^2 is the unit disk, both inside $\mathbb{R}^2$.) Compute the homology groups of X with integer coefficients.

(Q-10) Consider the following subsets of $\mathbb{R}^3$:

$$Z = \{(0,0,z) \mid z \in \mathbb{R}\}$$
$$C_1 = \{(\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\}$$
$$C_2 = \{(2+\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\}.$$

Prove that there is no self-homeomorphism of $\mathbb{R}^3$ that takes $Z \cup C_1$ to $Z \cup C_2$.