Qualifying Examination
LOGIC
Fall 2021

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Say that $\mathcal{M} \models T$ is *minimal* if $\mathcal{M}$ has no proper elementary submodels.

(1a) Give an example of a theory with a prime model that is not minimal.

(1b) Show that if a complete theory has a prime model and a minimal model, then they are isomorphic.

(1c) Show that $\mathrm{Th}(\mathbb{Z}, +)$ has a minimal model which is not prime.

Problem 2. Prove that the field of reals $(\mathbb{R}, +, \times, 0, 1)$ is not interpretable in the field of complex numbers $(\mathbb{C}, +, \times, 0, 1)$.

You may use Tarski's theorems that $(\mathbb{C}, +, \times, 0, 1)$ and $(\mathbb{R}, +, \times, 0, 1, <)$ eliminate quantifiers.

Problem 3. For an $\mathcal{L}_{\mathrm{ar}}$-sentence σ, let $\Box\sigma := \Box_{\mathrm{PA}}\sigma$. We define inductively $\Box^0\sigma := \sigma$ and $\Box^{n+1}\sigma := \Box(\Box^n\sigma)$.

In the following, assume that PA is consistent.

(3a) Show that there is a formula $\theta(x, y)$ such that for all sentences σ and all $n > 0$, $\mathrm{PA} \vdash \theta(\underline{n}, \#\sigma) \leftrightarrow \Box^n\sigma$. Here $\underline{n}$ is the numeral representing n, and $\#\sigma$ is the Gödel numeral of σ.

(3b) Is there a formula $\theta(x, y)$ with the property in (1) satisfied for all σ and n, including $n = 0$?

Problem 4. Prove that the theory of $(\mathbb{N}; 0, 1, +, |)$ is undecidable, where $|$ is the binary relation of divisibility on $\mathbb{N}$.

Problem 5. Assume $V = L$. Prove that there is a sequence of structures $(L_{\beta_\alpha}; \in, A_\alpha)$, for $\alpha < \omega_2$, where $\alpha \leq \beta_\alpha < \omega_2$ and $A_\alpha \subseteq \beta_\alpha$, so that for every $A \subseteq \omega_3$, there is H so that $(H; \in, A \cap H) \preceq (L_{\omega_3}; \in, A)$, $H \cap \omega_2$ is equal to an ordinal $\alpha < \omega_2$, and $(H; \in, A \cap H)$ is isomorphic to $(L_{\beta_\alpha}; \in, A_\alpha)$.

Problem 6. Say that L_γ is *definably regular*, if there is no cofinal $f\colon L_\alpha \to \gamma$ with $\alpha < \gamma$ and f first order definable with parameters over L_γ. Suppose $\gamma > \omega$ is a limit ordinal and L_γ is definably regular. Prove that L_γ satisfies ZF-Powerset.

Problem 7. Let A be a recursive set so that both A and its complement A^c are infinite. Classify the following in the arithmetic hierarchy. If you work by reducing to sets of known complexity, you must prove the classification results for these sets. Here $\varphi_e[A]$ is the image of A under the eth partial recursive function.

(7a) $\{e \mid \varphi_e[A] \subseteq A \wedge \varphi_e[A^c] \subseteq A^c\}$.
(7b) $\{e \mid \varphi_e[A] = A \wedge \varphi_e[A^c] = A^c\}$.

Problem 8. As usual, φ_e is the eth partial recursive function, and e is said to be a code for the function φ_e. Let r be a wellordering on ω. For each n, let I_n be the r-initial segment of ω consisting of all i which are r-below n. Let P on $\omega \times \omega$ be a function so that: if n is the r-minimal element, then for all i, $P(n,i) = n$; if n is an r-successor, then for all i, $P(n,i)$ is the r-predecessor of n; and if n is an r-limit, then $\{P(n,i) \mid i < \omega\}$ is an r-cofinal subset of I_n. Suppose that P is recursive.

Let h be recursive and total. Prove that there is a total recursive f so that for every n, $f(n)$ is equal to $h(e_n)$ for some code e_n for $f \restriction I_n$.