Qualifying Exam, Fall 2021

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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[1] **(10 Pts.)**
(a) (6 Pts.) Find the critical points (points satisfying the Lagrange condition) and local extremizers of

$$\begin{aligned} & f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3 \\ & \text{subject to} \quad x_1^2 + x_2^2 + x_3^2 = 4. \end{aligned}$$

Explain how these values are found.

(b) (4 Pts.) Find the solutions to

$$\begin{aligned} & \text{maximize} \quad x^\top \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} x \\ & \text{subject to} \quad \|x\| = 1. \end{aligned}$$

Explain how these values are found.

[2] (10 Pts.) Let

$$B := \begin{bmatrix} I & A \\ A^* & I \end{bmatrix},$$

where A is square with $\|A\|_2 \leq 1$ and I denotes the identity matrix of the same size. Prove that the condition number of B (recall $\kappa(B) = \|B\|_2 \|B^{-1}\|_2$ is the ratio of the largest to smallest singular value) satisfies

$$\kappa(B) = \frac{1 + \|A\|_2}{1 - \|A\|_2}.$$

Hint: Consider the SVD of $A = U\Sigma V^*$ and simplify the product of B times $\begin{bmatrix} U \\ \pm V \end{bmatrix}$.

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[3] (10 Pts.) Let $A \in \mathbb{R}^{n \times n}$. Recall that the Jacobi iteration on $Ax = b$ is given by

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b, \quad M = D, \quad N = -(L + U),$$

where L, D, U are lower triangular, diagonal, and upper triangular parts of A. Introducing a positive weight ω, the Jacobi Over-Relaxation (JOR) iteration uses

$$M = \frac{1}{\omega} D, \quad N = -\left(\left(1 - \frac{1}{\omega}\right) D + L + U \right).$$

Prove that if Jacobi iteration converges, JOR would also converge when $0 < \omega \leq 1$.

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[4] **(10 Pts.)** Recall that the Lanczos iteration tridiagonalizes a hermitian A by building towards

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

The Lanczos algorithm is given by

$$\begin{aligned}
&\beta_0 = 0, \quad q_0 = 0, \quad b = \text{arbitrary}, \quad q_1 = b/\|b\| \\
&\text{for } n = 1, 2, 3, \ldots \\
&\qquad v = Aq_n \\
&\qquad a_n = q_n^T v \\
&\qquad v = v - \beta_{n-1} q_{n-1} - \alpha_n q_n \\
&\qquad \beta_n = \|v\| \\
&\qquad q_{n+1} = v/\beta_n
\end{aligned}$$

(a) (6 Pts.) Show that if a symmetric real matrix A has a multiple eigenvalue, then the Lanczos iteration terminates prematurely.

(b) (4 Pts.) Show whether the converse is true (premature termination $\Rightarrow$ multiple eigenvalue).

[5] (10 Pts.) Let A be a positive definite symmetric $n \times n$ matrix and consider the system of equations $Ax = b$. Let $\{z_1, z_2, \ldots, z_n\}$ be a set of A-orthogonal non-zero vectors (recall A-orthogonality means z_i and Az_j are orthogonal for all $i \neq j$). Given an initial starting point x_0, define the conjugate directions iteration by

$$w_k = \frac{\langle z_k, b - Ax_{k-1} \rangle}{\langle z_k, Az_k \rangle}, \quad x_k = x_{k-1} + w_k z_k.$$

Prove (assuming exact arithmetic) that $Ax_n = b$.

[6] (10 Pts.) Let $A = QR$ be a reduced QR factorization for a tall matrix A of size $N \times n$ ($N \geq n$). Prove that if R has m nonzero values on its diagonal then $\text{rank}(A) \geq m$.

[7] (10 Pts.) Recall that GMRES for $Ax = b$ (with $A \in \mathbb{R}^{m\times m}$) combines the Arnoldi algorithm with a least squares solve:

$$
\begin{aligned}
&q_1 = b/\|b\| \\
&\text{for } n = 1, 2, 3, \ldots \\
&\quad v = Aq_n \\
&\quad \text{for } j = 1, 2, \ldots, n \\
&\quad\quad h_{jn} = q_j^* v \\
&\quad\quad v = v - h_{jn} q_j \\
&\quad h_{n+1,n} = \|v\| \\
&\quad q_{n+1} = v/h_{n+1,n} \\
&\quad x_n = \min_{x\in\mathcal{K}_n} \|Ax - b\|_2
\end{aligned}
$$

where $\mathcal{K}_n = \langle b, Ab, \ldots, A^{n-1}b\rangle$. Assume that A is nonsingular and $b \neq 0$. Assume exact arithmetic. Suppose that at iteration n (with $n < m$) we encounter $h_{n+1,n} = 0$ ("Arnoldi breakdown").

(a) (4 Pts.) Show that $A\mathcal{K}_n \subseteq \mathcal{K}_n$. (Hint: Arnoldi iteration defines a recurrence $Aq_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}$.)

(b) (4 Pts.) Show that this encounter guarantees that the solution x satisfies $x \in \mathcal{K}_n$ so that $Ax = b$ is solved.

(c) (2 Pts.) Assuming you are given a diagonalizable matrix A and a number $n < m$. Describe (with explanation) a strategy of constructing a b vector so that the breakdown will occur no later than step n.

[8] (10 Pts.)
(a) (4 Pts.) Consider the algorithm $x^{(k+1)} = x^{(k)} - \alpha_k M_k \nabla f(x^{(k)})$, where $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1$, $M_k = \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix}$ with $a \in \mathbb{R}$, and

$$\alpha_k = \operatorname{argmin}_{\alpha \geq 0} f(x^{(k)} - \alpha M_k \nabla f(x^{(k)})).$$

Suppose that at some iteration k we have $\nabla f(x^{(k)}) = [1, -1]^\top$. Find the largest range of values of a that guarantees $\alpha_k > 0$.

(b) (6 Pts.) Consider a function $f\colon \mathbb{R}^d \to \mathbb{R}$ with $f(w) \geq c$ for all $w \in \mathbb{R}^d$. Assume there is some $L > 0$ so that the *descent lemma holds*:

$$f(w') \leq f(w) + \nabla f(w)^\top (w' - w) + \frac{L}{2}\|w' - w\|^2 \quad \text{for all } w', w \in \mathbb{R}^d.$$

Show that there exists $\alpha \in \mathbb{R}$ such that if we run gradient descent with fixed step size α (i.e., we create a sequence $w^{(t+1)} = w^{(t)} - \alpha \nabla f(w^{(t)})$ with some initial value $w^{(0)}$), then

$$\min_{0 \leq t \leq T-1} \|\nabla f(w^{(t)})\|^2 \leq \frac{2L}{T} |f(w^{(0)}) - c|.$$

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