(a) Find the critical points (points satisfying the Lagrange condition) and local extremizers of

\[ f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3 \]
subject to

\[ x_1^2 + x_2^2 + x_3^2 = 4. \]

Explain how these values are found.

(b) Find the solutions to

\[
\begin{align*}
\text{maximize} & \quad x^\top \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} x \\
\text{subject to} & \quad \|x\| = 1.
\end{align*}
\]

Explain how these values are found.
Let \( B := \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \),

where \( A \) is square with \( \|A\|_2 \leq 1 \) and \( I \) denotes the identity matrix of the same size. Prove that the condition number of \( B \) (recall \( \kappa(B) = \|B\| \|B^{-1}\|_2 \) is the ratio of the largest to smallest singular value) satisfies

\[
\kappa(B) = 1 + \frac{\|A\|_2}{1 - \|A\|_2}.
\]

Hint: Consider the SVD of \( A = U\Sigma V^* \) and simplify the product of \( B \) times \( \begin{bmatrix} U \\ \pm V \end{bmatrix} \).
[3] (10 Pts.) Let $A \in \mathbb{R}^{n \times n}$. Recall that the Jacobi iteration on $Ax = b$ is given by

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad M = D, \quad N = -(L + U),$$

where $L, D, U$ are lower triangular, diagonal, and upper triangular parts of $A$. Introducing a positive weight $\omega$, the Jacobi Over-Relaxation (JOR) iteration uses

$$M = \frac{1}{\omega}D, \quad N = -\left(\left(1 - \frac{1}{\omega}\right)D + L + U\right).$$

Prove that if Jacobi iteration converges, JOR would also converge when $0 < \omega \leq 1$. 
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[4] (10 Pts.) Recall that the Lanczos iteration tridiagonalizes a hermitian $A$ by building towards

$$
T_n = \begin{pmatrix}
\alpha_1 & \beta_1 & & \\
\beta_1 & \alpha_2 & \beta_2 & \\
& \ddots & \ddots & \ddots \\
& & \beta_{n-1} & \alpha_n
\end{pmatrix}
$$

The Lanczos algorithm is given by

$$
\beta_0 = 0, \quad q_0 = 0, \quad b = \text{arbitrary}, \quad q_1 = b/\|b\|
$$

for $n = 1, 2, 3, \ldots$

$$
v = Aq_n \\
a_n = q_n^Tv \\
v = v - \beta_{n-1}q_{n-1} - \alpha_nq_n \\
\beta_n = \|v\| \\
q_{n+1} = v/\beta_n
$$

(a) (6 Pts.) Show that if a symmetric real matrix $A$ has a multiple eigenvalue, then the Lanczos iteration terminates prematurely.

(b) (4 Pts.) Show whether the converse is true (premature termination $\Rightarrow$ multiple eigenvalue).
[5] (10 Pts.) Let $A$ be a positive definite symmetric $n \times n$ matrix and consider the system of equations $Ax = b$. Let $\{z_1, z_2, \ldots, z_n\}$ be a set of $A$-orthogonal non-zero vectors (recall $A$-orthogonality means $z_i$ and $Az_j$ are orthogonal for all $i \neq j$). Given an initial starting point $x_0$, define the conjugate directions iteration by

$$w_k = \frac{\langle z_k, b - Ax_{k-1}\rangle}{\langle z_k, Az_k\rangle}, \quad x_k = x_{k-1} + w_k z_k.$$ 

Prove (assuming exact arithmetic) that $Ax_n = b$. 
[6] (10 Pts.) Let $A = QR$ be a reduced QR factorization for a tall matrix $A$ of size $N \times n$ ($N \geq n$). Prove that if $R$ has $m$ nonzero values on its diagonal then $\text{rank}(A) \geq m$. 
Recall that GMRES for $Ax = b$ (with $A \in \mathbb{R}^{m \times m}$) combines the Arnoldi algorithm with a least squares solve:

$q_1 = \frac{b}{\|b\|}$

for $n = 1, 2, 3, \ldots$

$v = Aq_n$

for $j = 1, 2, \ldots, n$

$h_{jn} = q_j^*v$

$v = v - h_{jn}q_j$

$h_{n+1,n} = \|v\|$

$q_{n+1} = v/h_{n+1,n}$

$x_n = \min_{x \in K_n} \|Ax - b\|_2$

where $K_n = \langle b, Ab, \ldots, A^{n-1}b \rangle$. Assume that $A$ is nonsingular and $b \neq 0$. Assume exact arithmetic. Suppose that at iteration $n$ (with $n < m$) we encounter $h_{n+1,n} = 0$ (“Arnoldi breakdown”).

(a) (4 Pts.) Show that $AK_n \subseteq K_n$. (Hint: Arnoldi iteration defines a recurrence $Aq_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}$.)

(b) (4 Pts.) Show that this encounter guarantees that the solution $x$ satisfies $x \in K_n$ so that $Ax = b$ is solved.

(c) (2 Pts.) Assuming you are given a diagonalizable matrix $A$ and a number $n < m$. Describe (with explanation) a strategy of constructing a $b$ vector so that the breakdown will occur no later than step $n$. 
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[8] (10 Pts.)
(a) (4 Pts.) Consider the algorithm \( x^{(k+1)} = x^{(k)} - \alpha_k M_k \nabla f(x^{(k)}) \), where \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, f \in C^1 \), \( M_k = \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix} \) with \( a \in \mathbb{R} \), and

\[
\alpha_k = \arg \min_{\alpha \geq 0} f(x^{(k)}) - \alpha M_k \nabla f(x^{(k)}).
\]

Suppose that at some iteration \( k \) we have \( \nabla f(x^{(k)}) = [1, -1]^\top \). Find the largest range of values of \( a \) that guarantees \( \alpha_k > 0 \).

(b) (6 Pts.) Consider a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) with \( f(w) \geq c \) for all \( w \in \mathbb{R}^d \). Assume there is some \( L > 0 \) so that the descent lemma holds:

\[
f(w') \leq f(w) + \nabla f(w)^\top (w' - w) + \frac{L}{2} \|w' - w\|^2 \quad \text{for all } w', w \in \mathbb{R}^d.
\]

Show that there exists \( \alpha \in \mathbb{R} \) such that if we run gradient descent with fixed step size \( \alpha \) (i.e., we create a sequence \( w^{(t+1)} = w^{(t)} - \alpha \nabla f(w^{(t)}) \) with some initial value \( w^{(0)} \)), then

\[
\min_{0 \leq t \leq T-1} \|\nabla f(w^{(t)})\|^2 \leq \frac{2L}{T} |f(w^{(0)}) - c|.
\]