Qualifying Exam, Fall 2021

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let $P_{0,1,\ldots,n} := P_{x_0,x_1,\ldots,x_n}$ be the interpolating Lagrange polynomial of degree at most n through the points $x_0, x_1, ..., x_n$ and values $f(x_0)$, ..., $f(x_n)$, such that $P_{0,1,\ldots,n}(x_i) = f(x_i)$.

(a) Let $i, j \in \{0, 1, ..., n\}$ be two distinct integers. Express $P_{0,1,\ldots,n}$ in terms of $P_{0,\ldots,i-1,i+1,\ldots,n}$ and $P_{0,\ldots,j-1,j+1,\ldots,n}$. Justify your answer.

(b) Suppose $x_j = j$ for $j = 0, 1, 2, 3$ and it is known that $P_{0,1}(x) = 5x - 1$, $P_{1,2}(x) = 3x + 1$, and $P_{1,2,3}(1.5) = 4$. Find $P_{0,1,2,3}(1.5)$.

[2] (5 Pts.) The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value of 6, and Simpson's rule gives the value 3. What is $f(1)$?

[3] (5 Pts.) Consider the two point boundary value problem determining $u(x)$ for $x \in [0, 1]$,

$$\Delta u = f \qquad u(0) = u(1) = 0$$

(a) If Δ_h is the three point difference approximation to Δu on a uniform mesh with mesh width h i.e.

$$\Delta u \approx \Delta_h u = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

let $\tilde{u}$ be the solution of the equations obtained by solving the difference equations arising from the approximation

$$\Delta_h \tilde{u} = f + \frac{h^2}{12} f_{xx} \qquad \tilde{u}(0) = \tilde{u}(1) = 0 \quad (1)$$

Assuming $f(x)$ is smooth and f_{xx} is known, what is the order of the trucation error associated with this approximation?

(b) Assuming f_{xx} is not known, give the difference approximation to f_{xx} that, when incorporated into (1), results in an approximation for u that has the same order of local truncation error as when f_{xx} is known. Justify your answer.

[4] (5 Pts.) Consider using Euler's method with a fixed timestep k to determine approximation solutions of linear constant coefficient system of ODE's

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \mathrm{A}\vec{u} \qquad \vec{u}(0) = u_0$$

for $t \in [0, \mathrm{T}]$ and A is a diagonalizable real N $\times$ N matrix.

(a) What is the restriction on the timestep, k, that arises from a consideration of the region of absolute stability for Euler's method?

(b) If $\mathrm{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ will the restriction ever be satisfied?

(c) Will Euler's method produce a convergent solution as $k \to 0$ over an interval $[0, \mathrm{T}]$ when applied to the system with the matrix in (b)? Explain.

[5] (10 Pts.) Consider a numerical method of the form

$$y^{n+1} = y^n + k\,\beta_1\,f(y^n) + k\,\beta_2 f(y^{n+p})$$

with p possibly being non-integer ($y^{n+p} = y(t_n + p\,k)$) for the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \qquad y(0) = y_0$$

over the interval $[0, \mathrm{T}]$ with uniform timestep k and $f(y)$ smooth.

(a) Assuming that y^{n+p} is known, determine the unique values of β_1, β_2 and p that result in a method with highest local truncation error. Show your work.

(b) If Euler's method is used to predict the value of y^{n+p} needed to implement the method in (a), what is the order of the resulting one-step method? Justify your answer.

(c) Give a derivation of an error bound for the method in (b) assuming $f(y)$ is smooth and has a global Lipschitz constant K.

[6] (10 Pts.) **(a)** Consider the equation

$$u_t + a\, u_x = u_{xx}$$

to be solved for $t > 0$, $0 < x < 1$, u periodic in x with period 1, and $a > 0$.

(a) Construct an explicit finite difference scheme which is consistent and satisfies a maximum principle when

$$a\frac{dt}{dx} + 2\frac{dt}{dx^2} \leq 1$$

(b) Next consider the consistent scheme:

$$u_i^{n+1} = u_i^n - \frac{a}{2}\frac{dt}{dx}\left(u_{i+1}^n - u_{i-1}^n\right) - \frac{dt}{dx^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Find criteria on a, dt, dx which guarantee a maximum principle for this scheme. Which of these schemes do you prefer? Explain

[7] (10 Pts.) Consider the second order partial differential equation

$$u_{tt} + 2bu_{tx} - a^2u_{xx} + cu_x + du_t = 0$$

where a, b, c and d are real numbers, to be solved for $t > 0$, $0 < x < 1$, u periodic in x with period 1 and initial data $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.

(b) For what values of a, b, c, and d is the problem well posed?

(b) Devise a convergent finite difference scheme for the well posed problems. Justify your answers,

[8] (10 Pts.) Consider the problem,

$$\begin{aligned} -\text{div}\Big(a(x)\nabla u\Big) + b(x)u &= f(x), \quad && x = (x_1, x_2) \in \Omega, \\ u &= 0, && x \in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 4, && x \in \partial\Omega_2, \end{aligned}$$

where $\Omega = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}$,

$\partial\Omega_1 = \{x|\ x_1 = 0,\ 0 \leq x_2 \leq 1\} \cup \{x|\ x_2 = 0,\ 0 \leq x_1 \leq 1\}$,

$\partial\Omega_2 = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$,

$0 < a \leq a(x) \leq A$, $0 < b \leq b(x) \leq B$, with a and b smooth functions and $f \in L^2(\Omega)$.

(a) Find the weak variational formulation and show that the problem is well-posed by verifying the assumptions of the Lax-Milgram Lemma and by analyzing the appropriate linear and bilinear forms.

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.