DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

[1] (5 Pts.) Let \( P_{0,0,\ldots,n} := P_{x_0,x_1,\ldots,x_n} \) be the interpolating Lagrange polynomial of degree at most \( n \) through the points \( x_0, x_1, \ldots, x_n \) and values \( f(x_0), \ldots, f(x_n) \), such that \( P_{0,0,\ldots,n}(x_i) = f(x_i) \).

(a) Let \( i, j \in \{0, 1, \ldots, n\} \) be two distinct integers. Express \( P_{0,0,\ldots,n} \) in terms of \( P_{0,0,\ldots,i-1,i+1,\ldots,n} \) and \( P_{0,0,\ldots,j-1,j+1,\ldots,n} \). Justify your answer.

(b) Suppose \( x_j = j \) for \( j = 0, 1, 2, 3 \) and it is known that \( P_{0,0,\ldots,1}(x) = 5x - 1 \), \( P_{1,1,\ldots,2}(x) = 3x + 1 \), and \( P_{1,1,\ldots,3}(1.5) = 4 \). Find \( P_{0,0,\ldots,3}(1.5) \).

[2] (5 Pts.) The Trapezoidal rule applied to \( \int_0^2 f(x)\,dx \) gives the value of 6, and Simpson’s rule gives the value 3. What is \( f(1) \) ?

[3] (5 Pts.) Consider the two point boundary value problem determining \( u(x) \) for \( x \in [0, 1] \),

\[
\Delta u = f \quad u(0) = u(1) = 0
\]

(a) If \( \Delta_h \) is the three point difference approximation to \( \Delta u \) on a uniform mesh with mesh width \( h \) i.e.

\[
\Delta u \approx \Delta_h u = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}
\]

let \( \tilde{u} \) be the solution of the equations obtained by solving the difference equations arising from the approximation

\[
\Delta_h \tilde{u} = f + \frac{h^2}{12} f_{xx} \quad \tilde{u}(0) = \tilde{u}(1) = 0 \quad (1)
\]

Assuming \( f(x) \) is smooth and \( f_{xx} \) is known, what is the order of the truncation error associated with this approximation?

(b) Assuming \( f_{xx} \) is not known, give the difference approximation to \( f_{xx} \) that, when incorporated into (1), results in an approximation for \( u \) that has the same order of local truncation error as when \( f_{xx} \) is known. Justify your answer.
Consider using Euler’s method with a fixed timestep $k$ to determine approximation solutions of linear constant coefficient system of ODE’s

$$\frac{d\vec{u}}{dt} = A\vec{u} \quad \vec{u}(0) = \vec{u}_0$$

for $t \in [0, T]$ and $A$ is a diagonalizable real $N \times N$ matrix.

(a) What is the restriction on the timestep, $k$, that arises from a consideration of the region of absolute stability for Euler’s method?

(b) If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ will the restriction ever be satisfied?

(c) Will Euler’s method produce a convergent solution as $k \to 0$ over an interval $[0, T]$ when applied to the system with the matrix in (b)? Explain.

Consider a numerical method of the form

$$y^{n+1} = y^n + k \beta_1 f(y^n) + k \beta_2 f(y^{n+p})$$

with $p$ possibly being non-integer ($y^{n+p} = y(t_n + pk)$) for the initial value problem

$$\frac{dy}{dt} = f(y) \quad y(0) = y_0$$

over the interval $[0, T]$ with uniform timestep $k$ and $f(y)$ smooth.

(a) Assuming that $y^{n+p}$ is known, determine the unique values of $\beta_1$, $\beta_2$ and $p$ that result in a method with highest local truncation error. Show your work.

(b) If Euler’s method is used to predict the value of $y^{n+p}$ needed to implement the method in (a), what is the order of the resulting one-step method? Justify your answer.

(c) Give a derivation of an error bound for the method in (b) assuming $f(y)$ is smooth and has a global Lipschitz constant $K$. 
[6] (10 Pts.) (a) Consider the equation
\[ u_t + a u_x = u_{xx} \]
to be solved for \( t > 0, 0 < x < 1, \) \( u \) periodic in \( x \) with period 1, and \( a > 0. \)

(a) Construct an explicit finite difference scheme which is consistent and satisfies a maximum principle when
\[ a \frac{dt}{dx} + 2 \frac{dt}{dx^2} \leq 1 \]

(b) Next consider the consistent scheme:
\[ u_{i}^{n+1} = u_{i}^{n} - a \frac{dt}{dx} \left( u_{i+1}^{n} - u_{i-1}^{n} \right) - \frac{dt}{dx^2} \left( u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n} \right) \]

Find criteria on \( a, dt, dx \) which guarantee a maximum principle for this scheme. Which of these schemes do you prefer? Explain

[7] (10 Pts.) Consider the second order partial differential equation
\[ u_{tt} + 2bu_{tx} - a^2u_{xx} + cu_x + du_t = 0 \]
where \( a, b, c \) and \( d \) are real numbers, to be solved for \( t > 0, 0 < x < 1, \) \( u \) periodic in \( x \) with period 1 and initial data \( u(x,0) = f(x), \) \( u_t(x,0) = g(x). \)

(b) For what values of \( a, b, c, \) and \( d \) is the problem well posed?

(b) Devise a convergent finite difference scheme for the well posed problems. Justify your answers,

[8] (10 Pts.) Consider the problem,
\[ -\text{div}(a(x)\nabla u) + b(x)u = f(x), \quad x = (x_1, x_2) \in \Omega, \]
\[ u = 0, \quad x \in \partial \Omega_1, \]
\[ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 4, \quad x \in \partial \Omega_2, \]

where \( \Omega = \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}, \)
\( \partial \Omega_1 = \{x \mid x_1 = 0, 0 \leq x_2 \leq 1\} \cup \{x \mid x_2 = 0, 0 \leq x_1 \leq 1\}, \)
\( \partial \Omega_2 = \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 = 1\}, \)
\( 0 < a(x) \leq A, 0 < b(x) \leq B, \) with \( a \) and \( b \) smooth functions and \( f \in L^2(\Omega). \)

(a) Find the weak variational formulation and show that the problem is well-posed by verifying the assumptions of the Lax-Milgram Lemma and by analyzing the appropriate linear and bilinear forms.

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.