Algebra Qualifying Exam, Fall 2021

Instructions: Please do the following ten problems. Write your UID number only, not your name. All answers must be justified. State clearly theorems that you use.

1. Let $a \in \mathbb{Q}$ and $b, d \in \mathbb{Q}^\times$, and suppose that d is not a cube in $\mathbb{Q}^\times$. Find the minimal polynomial of $a + b\sqrt[3]{d}$ over $\mathbb{Q}$.

2. Let K be a field, and consider the ring $R = K[x]/(x^2)$. Show that every free submodule N of an R-module M is a direct summand of M.

3. Show that there are no simple groups of order $24p$, where p is a prime number greater than 11.

4. Let G be a cyclic group of order 12. For each of the fields $F = \mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, write the regular representation $F[G]$ as a direct sum of simple (i.e., irreducible) modules.

5. Consider a sequence of sets S_i for $i \geq 0$ and maps $\phi_i \colon S_i \to S_{i-1}$ for $i \geq 1$. Suppose that there exists a positive integer N such that the orders of the images of the maps ϕ_i are bounded above by N. Show that $\varprojlim_i S_i$ is finite.

6. Consider the elements $g = (12)$ and $h = (23)$ in the symmetric group S_3. Consider the action of S_3 on the polynomial ring $\mathbb{C}[x, y]$ determined by $g(x) = y$, $g(y) = x$, $h(x) = x - y$, and $h(y) = -y$. (Here S_3 is acting on $\mathbb{C}[x, y]$ as a $\mathbb{C}$-algebra. You need not check that this action is well-defined.) Let V be the complex vector space of homogeneous polynomials of degree 3 in x and y; this is mapped into itself by S_3. Compute the character of V. When V is written as a direct sum of irreducible representations of S_3, find the number of times each irreducible representation of S_3 occurs.

7. Define commutative $\mathbb{Q}$-algebras $A = \mathbb{Q}$, $B = \mathbb{Q}[x]$, and $C = \mathbb{Q}[x]/(x(x-1))$. Let $A \to C$ and $B \to C$ be the unique $\mathbb{Q}$-algebra homomorphisms such that x in B maps to x in C. Describe the pullback (also called "fiber product") $R = A \times_C B$ in the category of commutative $\mathbb{Q}$-algebras, as the quotient by an explicit ideal of the polynomial ring over $\mathbb{Q}$ on some set of generators. Is R noetherian?

8. Let A be a commutative ring and T an A-module. Define a functor from A-modules to A-modules by $F(M) = M \otimes_A T$. What is the right adjoint functor of F? Show that if F has a left adjoint, then T must be a flat A-module, and also a finitely generated A-module.

9. The *outer automorphism group* of a group H is the quotient of the group of automorphisms of H by the subgroup of inner automorphisms. It is known that the outer automorphism group of every finite simple group is solvable. Using that, show that if G is a finite group with a normal subgroup N such that both N and G/N are nonabelian simple groups, then G is isomorphic to the product group $N \times (G/N)$.

10. Let R_1 and R_2 be rings (not necessarily commutative), and let M be an (R_1, R_2)-bimodule. (That is, M is an abelian group which is a left R_1-module and a right R_2-module, and the two actions commute.) Then the matrices

$$\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$$

form a ring R, by the usual formulas for matrix addition and multiplication. Compute the Jacobson radical of R in terms of M and the Jacobson radicals of R_1 and R_2.