THE ANALYSIS QUALIFYING EXAM

The analysis qualifying exam ("analysis qual") is the shared responsibility of the analysis, functional analysis, and probability & mathematical physics research groups. It corresponds to the graduate course sequences 245ABC and 246AB, with only a few results taken from 245C.

The following sections describe the prerequisites from undergraduate study and then the graduate-level topics for this exam. Students are encouraged to study these topics broadly, but will have some ability to select problems on the exam: the analysis qual usually poses six questions each on real and complex analysis, and three essentially complete solutions (9/10 or above) in each of these two areas will usually lead to a Pass. Scores that lie close to this threshold or that depend heavily on partial credit are assessed on a case-by-case basis. Note that an excellent score on real analysis but a poor score on complex analysis, or the reverse, is usually not enough for a Pass.

1. Prerequisites

The analysis qual and corresponding courses will assume a good working knowledge of analysis at about the undergraduate honors level. This is taken to include all analysis topics for the Basic Exam, and also the following material from point-set topology:

- the main classes of topological spaces (Hausdorff, compact, etc.) and their basic properties
- continuous functions and their basic properties
- Urysohn's lemma and the Tietze extension theorem
- product topologies and Tychonoff's theorem

These prerequisites will not generally be taught again. Each of these topics is covered in most of the real analysis references suggested below, or in Chapters 1 and 2 of Gamelin and Greene, *Introduction to Topology*. Students who still need to master these topics are encouraged to approach a 245AB instructor or any other member of the relevant research groups for support or guidance. In the past, many students in this position have benefited from studying these topics as a group.

2. The ceiling/floor principle

Beyond the prerequisites described above, the topic lists in the next section fix the relationship between the qual and its dedicated courses according to a "ceiling/floor principle":

> **Ceiling**: The background knowledge relevant to the exam *must not* exceed this list.

Floor: This list of topics *must* be covered in the relevant courses.
The instructor may cover additional topics if time permits.

Warning: This principle does not mean that qual questions are generally comparable to final-exam questions from 245ABC and 246AB. On the whole, qual questions require more synthesis of ideas from several parts of these courses, and offer less indication of which results may be relevant. Students preparing for the qual are stongly encouraged to study the topics thematically and also practice on past exam problems. Some past exam problems are no longer representative of the qual, so students should seek faculty or TA guidance on their selection.

3. Topic lists

The order of topics in each list below is merely a suggestion. It may be altered at the instructor's discretion, as long as all topics are covered.

3.1. Real Analysis.

The first half of the analysis qual covers real analysis. The relevant course sequence is **245ABC**. In practice, the dependence on 245C is limited to a few results from the first half of that course.

Topics.

General measure theory (often 245A)

- Construction and main properties of Lebesgue measure, including in several dimensions
- Abstract measure spaces and their basic properties
- General approaches to constructing measures such as the Carathéodory–Hahn approach
- Lebesgue integration
- Monotone and dominated convergence theorems and Fatou's lemma
- Lusin's and Egorov's theorems
- Product measures and Fubini's theorem
- Signed measures, including the Hahn and Jordan decompositions
- Abstract differentiation of measures and Radon–Nikodym derivatives

Measure theory on Euclidean or other topological spaces (often 245A)

- Lebesgue–Stieltjes measures on the real line
- Change of variables for the Lebesgue integral on Euclidean spaces
- Covering lemmas, the Hardy–Littlewood maximal function, and the Lebesgue differentiation theorem on Euclidean spaces; the Lebesgue decomposition of a measure; functions of bounded variation on the real line and the fundamental theorem of calculus for the Lebesgue integral
- Regularity properties of Borel measures on metric or topological spaces

Functional analysis (often 245B)

- Banach spaces: basic definitions and properties
- Linear functionals and the Hahn–Banach theorem; dual spaces

- Weak and weak* topologies, including the Banach–Alaoglu theorem (*may* be included in a broader treatment of locally convex spaces)
- Important examples: L^p-spaces, Hölder's inequality, Jensen's inequality, and the Riesz representation theorems for $(L^p)^*$
- Important examples: $C(X)$, including the Arzelà–Ascoli and Stone–Weierstrass theorems (revisit from the Basic Exam) and the Riesz representation theorem for $C(X)^*$
- Hilbert spaces, including the basic geometry of orthogonality, orthonormal bases, and the associated Riesz representation theorem

Fourier and harmonic analysis (often 245B and first parts of 245C)

- Convolutions, smoothing of functions, and approximate identities
- The use of distribution functions for L^p-functions; weak-L^p spaces
- Trigonometric series, including the Hilbert-space point of view
- Definition and basic properties of the Fourier transform in Euclidean spaces
- Parseval's inequality and Plancherel's theorem
- The Riemann–Lebesgue lemma

Suggested references. No single text corresponds exactly to the outline above, but many good texts cover these topics. The texts used for 245ABC are chosen by the individual instructors. Some popular choices are:

- R. Bass, *Real Analysis for Graduate Students*, available for purchase or online at `http://bass.math.uconn.edu/real.html`
- G.B. Folland, *Real Analysis: Modern Techniques and Their Applications* (2nd Ed. recommended)
- H.L. Royden, *Real Analysis* (3rd ed. recommended)
- W. Rudin, *Principles of Mathematical Analysis* (for some more elementary topics) and *Real and Complex Analysis* (for both real analysis and complex analysis topics)
- E.M. Stein and R. Shakarchi, *Real Analysis, Measure Theory, Integration and Hilbert Spaces* and *Fourier Analysis*
- R. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*

3.2. **Complex analysis.** The second half of the analysis qualifying exam covers complex analysis. The relevant course sequence is **246AB**.

Topics. Complex analysis (246AB)

- Holomorphic functions, Cauchy–Riemann equations, ∂- and $\bar{\partial}$-derivatives
- Path integrals and power series
- Goursat's lemma, Cauchy's theorem and Cauchy's integral formula
- Morera's theorem; Liouville's theorem
- Uniqueness theorem for analytic functions, maximum principle
- Open mapping theorem

- Elementary functions, exponential and logarithm functions in the complex plane
- Möbius transformations and the Riemann sphere
- Schwarz's lemma and the hyperbolic metric
- Winding numbers
- Isolated singularities and Laurent expansions; the Casorati–Weierstrass theorem
- The residue theorem; evaluation of definite integrals
- The argument principle and Rouché's theorem
- Locally uniform convergence and normal families
- Conformal mappings, Riemann mapping theorem, Schwarz reflection principle, Hurwitz's theorem
- Harmonic functions: conjugate functions, maximum principle, mean value property, Poisson integrals, Dirichlet problem for a disk, Harnack's principle, subharmonic functions
- Infinite products, series and product expansions, Mittag-Leffler theorem
- Analytic continuation and the monodromy principle
- Picard's theorem

Suggested references. No single text corresponds exactly to the outline above, but many good texts cover these topics. The texts used for 246ABC are chosen by the individual instructors. Some popular choices are:

- L. Ahlfors, *Complex Analysis*
- T.W. Gamelin, *Complex Analysis*
- W. Rudin, *Real and Complex Analysis* (for both real analysis and complex analysis topics)
- E.M. Stein and R. Shakarchi, *Complex Analysis*